Monotone utility convergence *

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Abstract

We show that the maximal expected utility satisfies a monotone continuity property with respect to increasing information: Let \((G^n_t)\) be a sequence of increasing filtrations converging to \((G^\infty_t)\), and \(u^n(x)\) and \(u^\infty(x)\) the maximal expected utility when investing on a financial market according to strategies adapted to \((G^n_t)\) and \((G^\infty_t)\) respectively. We give sufficient conditions for the convergence \(u^n(x) \to u^\infty(x)\) as \(n \to \infty\). We provide examples for which convergence does not hold.

In the second part we consider the utility based prices \(\pi^n\) and \(\pi^\infty\) of contingent claims under \((G^n_t)\) and \((G^\infty_t)\) respectively. We analyse to which extent \(\pi^n \to \pi^\infty\) as \(n \to \infty\).

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1 Introduction

The decisions of an investor on a financial market strongly depend on the information he has access to. Naturally the question arises, how the behaviour changes if the investor obtains additional information and how strong in average this change will be. Intuitively one may expect that if the information increases only slightly, then the optimal investment will not change much neither. Put differently, if the information converges, then the optimal investment and the related maximal expected utility will converge as-well. In

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this paper we aim at providing sufficient conditions for this convergence. One can interpret the monotone convergence property as information robustness of the financial market model we consider. In a second step we will look at the information dependence of utility based prices of contingent claims. We will analyse to which extent the prices satisfy a continuity property under increasing information.

We will model the information dependence of optimal investment by using different filtrations to which the investment strategies have to be adapted to. This technique has been widely used to model insiders on financial markets (see e.g. [7], [9]).

Here is a rough outline of the results: Let \((G^n_t)\) be a sequence of increasing filtrations and denote by \((G^\infty_t)\) the union. Let \(u^n(x)\) and \(u^\infty(x)\) be the supremum of the expected utility when investing on a continuous financial market according to standard sets of strategies adapted to \((G^n_t)\) and \((G^\infty_t)\) respectively. We will show under some weak assumptions, depending on the type of the utility function, that \(u^n(x)\) converges to \(u^\infty(x)\) as \(n \to \infty\).

Let \(u^n(x, B)\) and \(u^\infty(x, B)\) be the supremum of the expected utility if in addition there is a contingent claim \(B\) in the portfolio. Again under some natural conditions \(u^n(x, B)\) converges to \(u^\infty(x, B)\) as \(n \to \infty\). We provide examples for which convergence does not hold.

Finally we consider the utility based prices \(\pi_n\) and \(\pi_\infty\) of contingent claims under \((G^n_t)\) and \((G^\infty_t)\) respectively. We analyse to which extent \(\pi_n \to \pi_\infty\) as \(n \to \infty\).

2 Preliminaries

Let \((\Omega, \mathcal{F}, P)\) be a probability space, \(S : \Omega \times [0, T] \to \mathbb{R}\) a continuous stochastic process starting in zero. We interpret \(S\) as a price process and \(T\) as the time horizon. Suppose that \((\mathcal{F}_t)\) is a filtration satisfying the usual conditions. If \(S\) is a semimartingale with respect to \((\mathcal{F}_t)\), we denote by \(\mathcal{A}(\mathcal{F})\) the set of all \((\mathcal{F}_t)\)-predictable processes \(\theta\) which satisfy \(\theta_0 = 0\) and which are integrable with respect to \(S\) and \((\mathcal{F}_t)\) in the usual sense (see Protter [10]). The elements of \(\mathcal{A}(\mathcal{F})\) will be called strategies. Moreover, a strategy is called a-admissible if the stochastic integral process satisfies \((\theta \cdot S)_t \geq -a\), for all \(t \in [0, T]\). And more generally, \(\theta\) will be called admissible if it is \(a\)-admissible for some \(a\). Finally we say that \(S\) satisfies the no arbitrage condition (NA) with respect to \((\mathcal{F}_t)\) if there exists no admissible \(\theta \in \mathcal{A}(\mathcal{F})\) such that \((\theta \cdot S)_T \geq 0\) and \(P((\theta \cdot S)_T > 0) > 0\).

By a utility function \(U\) we mean any concave function \(U : \mathbb{R} \to \mathbb{R} \cup \{-\infty\}\).
The maximal expected utility with respect to \((\mathcal{F}_t)\) is defined by

\[
u^F(x) = \sup \{ EU(x + (\theta \cdot S)_T) : \theta \in \mathcal{A}(\mathcal{F}) \text{ is admissible} \}.
\]

We will also consider

\[
u^F_a(x) = \sup \{ EU(x + (\theta \cdot S)_T) : \theta \in \mathcal{A}(\mathcal{F}) \text{ is } a \text{- admissible} \}.
\]

Now let \((\mathcal{G}_n^\infty)\) be a sequence of increasing filtrations satisfying the usual conditions. Moreover we will suppose that \(S\) is a semimartingale relative to any \((\mathcal{G}_n^\infty)\). The smallest filtration satisfying the usual conditions and containing every filtration \((\mathcal{G}_n^\infty)\) will be denoted by

\[
\mathcal{G}_t^\infty = \bigcap_{s \geq t \wedge n \geq 1} \mathcal{G}_s^n.
\]

Throughout we suppose that \(S\) is a continuous \((\mathcal{G}_t^\infty)\)-semimartingale with decomposition

\[
S_t = M_t + \int_0^t \alpha_s \, d\langle M, M \rangle_s,
\]

where \(M\) is a \((\mathcal{G}_t^\infty)\)-local martingale starting in zero and \(\alpha\) a \((\mathcal{G}_t^\infty)\)-predictable process satisfying \(\int_0^T \alpha^2_t \, d\langle M, M \rangle_t < \infty\), a.s. Notice that some no arbitrage-type conditions, f.e. the (NFLVR) condition, imply the existence of such a semimartingale decomposition (see Ansel, Stricker [5], or Delbaen, Schachermayer [6]). Alternatively, if \(\lim_{x \to \infty} U(x) = \infty\) and \(u_{a}^\infty(x)\) is finite for some \(a > 0\), then there exists a decomposition of the form (1) (see [4], [2]).

Notice that also with respect to every subfiltration \((\mathcal{G}_n^\infty)\) we can find a decomposition of \(S = M^n + \alpha^n \cdot \langle M, M \rangle\) such that \(\alpha^n\) is locally square-integrable. This guarantees that for every \((\mathcal{G}_n^\infty)\)-measurable strategy \(\theta\) the stochastic integrals \((\theta \cdot \mathcal{G}^\infty S)\) and \((\theta \cdot \mathcal{G}^\infty S)\), defined with respect to \((\mathcal{G}_n^\infty)\) and \((\mathcal{G}_t^\infty)\) respectively, are the same. We therefore omit the filtrations in the definition of the integrals.

We denote by \(u^n(x)\) the maximal expected utility with respect to \((\mathcal{G}_n^\infty)\), and by \(\nu^\infty(x)\) the maximal expected utility with respect to \((\mathcal{G}_t^\infty)\). Similarly, we abbreviate \(u^n_a(x)\) and \(\nu^\infty_a(x)\) for \(a > 0\).

Since the sequence \((\mathcal{G}_n^\infty)\) is increasing, \(u^n(x)\) is increasing as-well. We will provide sufficient conditions for the convergence \(u^n(x) \to \nu^\infty(x)\) as \(n \to \infty\). We will have to distinguish between two types of our utility functions, depending on the so-called domain which is defined by \(\text{dom}(U) = \{y : U(y) > -\infty\}\). We will at first consider the case \(\text{dom}(U) = \mathbb{R}\), and then \(\text{dom}(U) \neq \mathbb{R}\).
3 Monotone utility convergence

3.1 Convergence in the case \( \text{dom}(U) = \mathbb{R} \)

Throughout this section we assume \( \text{dom}(U) = \mathbb{R} \).

We start with the observation that the utility maximum can be attained by using strategies in \( L^2(M) = \{ \theta \text{ measurable} : E \int_0^T \theta_t^2 d\langle M, M \rangle_t < \infty \} \).

For any filtration \((\mathcal{F}_t)\) we denote by \( L^2_{\mathcal{F}}(M) \) the set of all \((\mathcal{F}_t)\)-predictable processes \( \theta \in L^2(M) \).

Lemma 3.1. Let \((\mathcal{F}_t)\) be a filtration with respect to which \( S \) is a semi-martingale. Let \( x \in \mathbb{R} \) and \( a \in (0, \infty) \). Then

\[
u^\infty_a(x) = \sup \{ EU(x + (\theta \cdot S)_T) : \theta \in L^2_{\mathcal{F}}(M) \cap \mathcal{A}, (a - \varepsilon) - \text{adm. for some } \varepsilon > 0 \} \tag{2}\]

Proof. We prove at first for all \( x \in \mathbb{R} \)

\[
u^\infty(x) = \sup_{\varepsilon > 0} \sup \{ EU(x + (\theta \cdot S)_T) : \theta \in \mathcal{A}(G^\infty), (x - \varepsilon) - \text{adm.} \} \tag{3}\]

We have only to show that the LHS does not exceed the RHS. For this let \( \theta \in \mathcal{A}(G^\infty) \) such that \( EU(x + (\theta \cdot S)_T) > -\infty \). Put \( \theta^n = (1 - \frac{1}{n})\theta \) for all \( n \geq 1 \). Clearly, \( \theta^n \) is \((x - \frac{\varepsilon}{n})\)-admissible. Monotone convergence applied to the negative and positive part of \( U(x + (\theta^n \cdot S)_T) - U(x) \) implies

\[
\lim_n \ EU(x + (\theta^n \cdot S)_T) = \ EU(x + (\theta \cdot S)_T),
\]

and hence \( \nu^\infty(x) \) is smaller than RHS of equation (3).

Obviously the RHS of (2) is not bigger than the LHS. For the reverse inequality choose \( \varepsilon > 0 \) and an \((a - \varepsilon)\)-admissible strategy \( \theta \) satisfying \( EU(x + (\theta \cdot S)_T) > -\infty \). By equation (3) it is sufficient to show that \( EU(x + (\theta \cdot S)_T) \) is not greater than the RHS of (2). Since \( \theta \) is \( S \)-integrable, the stopping times

\[
T_n = T \wedge \inf \{ t \geq 0 : \int_0^t \theta^2_r \, d\langle M, M \rangle_r \leq n \}
\]

converge almost surely to \( T \) for \( n \to \infty \). Note that the strategies

\[
\theta^n = 1_{[0,T_n]}\theta
\]

are \((a - \varepsilon)\)-admissible and belong to \( L^2_{\mathcal{F}}(M) \). Fatou’s Lemma implies

\[
\lim \inf_n \ EU(x + (\theta^n \cdot S)_T) \geq \ EU(x + (\theta \cdot S)_T),
\]
and thus the result.

Let’s come back to the filtrations \((G^n_t)\) and \((G^\infty_t)\) introduced in section 2. We start by showing that the sequence \((u^n_a(x))\) satisfies a monotone convergence property.

**Theorem 3.2.** Let \(x \in \mathbb{R}\) and \(a \in (0, \infty)\). Then
\[
\lim_{n} u^n_a(x) = u^\infty_a(x).
\]

**Proof.** Let \(\theta \in L^2_{G^\infty}(M)\) be \((a - \varepsilon)\)-admissible. The stopping times
\[
\tau_k = T \land \inf\{t \geq 0 : \int_0^t \alpha_s^2 d\langle M, M \rangle_s \geq k\}
\]
converge to \(T\), a.s, and hence
\[
\lim inf_k E\left(U(x + (\theta \cdot S)_{\tau_k})\right) \geq E\left(U(x + (\theta \cdot S)_T)\right).
\]
By Lemma 3.1 it suffices to show that for all \(k \geq 1\), \(E\left(U(x + (\theta^n \cdot S)_{T_n})\right)\) is not greater than \(\sup_n u^n_a(x)\). To simplify notation we assume that \(T_n = T\) for some \(k\).

Now let \(\theta^n\) be the projection of \(\theta\) onto \(L^2_{G^n}\). Note that by Doob’s inequality there is a constant \(C > 0\), such that
\[
E((\theta^n - \theta) \cdot S)_T^2 \leq E((\theta^n - \theta) \cdot M)_T^2 + E((\theta^n - \theta) \alpha \cdot \langle M, M \rangle)_T^2.
\]
The first summand in the preceding line goes to 0, because \((\theta^n)\) converges to \(\theta\) in \(L^2(M)\). The second vanishes due to Kunita-Watanabe and due to our assumption that \(\int_0^T \alpha_s^2 d\langle M, M \rangle_s\) is bounded. Consequently, by choosing a subsequence if necessary, almost everywhere the sequence \((\theta^n \cdot S)\) converges uniformly to \((\theta \cdot S)\) on \([0, T]\). Now put
\[
T_n = T \land \inf\{t \geq 0 : (\theta^n \cdot S)_t \leq -a\}
\]
and \(\pi^n = 1_{[0, T_n]}(\theta^n)\). The strategies \(\pi^n\) are \(a\)-admissible and satisfy almost surely
\[
\lim_n (\pi^n \cdot S)_T = (\theta \cdot S)_T.
\]
With Fatou’s Lemma we obtain
\[
\lim inf_n E\left(U(x + (\pi^n \cdot S)_T)\right) \geq E\left(U(x + (\theta \cdot S)_T)\right),
\]
and hence the result. \(\square\)

We obtain immediately the following.
Corollary 3.3. For all $x \in \mathbb{R}$ we have

$$\lim_{n} u^{n}(x) = u^{\infty}(x).$$

3.2 Convergence in the case $\text{dom}(U) \neq \mathbb{R}$

Throughout this section we assume $\text{dom}(U) \neq \mathbb{R}$. To simplify notation we suppose that $\sup\{y : U(y) = -\infty\} = 0$. The analogue to Lemma 3.1 is the following.

Lemma 3.4. For $x > 0$ and $a \in (0, x]$ we have

$$u^{F}_{a}(x) = \sup\{EU(x + (\theta \cdot S)_{T}) : \theta \in L_{F}^{2}(M) \text{ and } (a-\varepsilon)-\text{adm. for some } \varepsilon > 0\}.$$

Proof. This can be shown like Lemma 3.1. Notice that Fatou’s lemma can only be applied if $a \leq x$. □

From this we can deduce the analogue to Theorem 3.2:

Theorem 3.5. For $x > 0$ and $a \in (0, x]$ we have

$$\lim_{n} u^{n}_{a}(x) = u^{\infty}_{a}(x).$$

Proof. Similar to the proof of Theorem 3.2. We only have to replace the stopping times $T_{n}$ by $T'_{n} = T \wedge \inf\{t > 0 : (\theta^{n} \cdot S)_{t} \leq -a + \frac{\varepsilon}{2}\}$. Then the strategies $\pi^{n} = 1_{[0,T'_{n}]}\theta^{n}$ are $(a-\frac{\varepsilon}{2})$ so that we can again apply Fatou’s lemma. □

We will see in Example 3.8 that Theorem 3.5 is not valid without the assumption that $a \in (0, x]$. However, we can skip it if $S$ satisfies the (NA) condition. To prove this we need

Lemma 3.6. Let $a > 0$ and $\theta \in \mathcal{A}(G^{\infty})$. Suppose $S$ satisfies the (NA) condition relative to $(G^{\infty})$. If $(\theta \cdot S)_{T} \geq -a$, a.s, then $\theta$ is $a$-admissible.

Proof. For every $\varepsilon > 0$ define a stopping time by

$$\tau_{\varepsilon} = \inf\{t > 0 : (\theta \cdot S)_{t} = -a - \varepsilon\} \wedge T.$$

Suppose $\theta$ is not $a$-admissible. Then for some $\varepsilon > 0$ we must have $P(\tau_{\varepsilon} < T) > 0$. The strategy $\pi = 1_{[\tau_{\varepsilon},T]}\theta$ satisfies $\pi \cdot S_{T} = 1_{[\tau_{\varepsilon},T]}[(\theta \cdot S)_{T} - (\theta \cdot S)_{\tau_{\varepsilon}}] \geq 0$, and $P((\pi \cdot S)_{T} > 0) = P(\tau_{\varepsilon} < T) > 0$. Hence $\pi$ is an arbitrage opportunity, a contradiction to (NA). □
Corollary 3.7. If $S$ satisfies (NA) relative to $(\mathcal{G}_t^\infty)$, then for all $x > 0$ we have

$$\lim_{n} u^n(x) = u^\infty(x).$$

Proof. Let $\theta$ be a $(\mathcal{G}_t^\infty)$-predictable strategy such that $EU(x + (\theta \cdot S)_T) > -\infty$. Then $(\theta \cdot S)_T \geq -x$, a.s. Lemma 3.6 implies that $\theta$ is $x$-admissible, and thus we have $u^\infty(x) = u^n(x)$. Similarly, we obtain $u^n(x) = u^n(x)$, $n \geq 1$.

The claim now follows from Theorem 3.5. $\square$

The next example shows that the assumption of (NA) is necessary in Corollary 3.7.

Example 3.8. Let $W$ be a Brownian motion with respect to $(\mathcal{F}_t)$ and suppose the price process is given by $S_t = S_0 + W_t$ with $S_0 > 0$ constant. Moreover let $(\psi_n)$ be an i.i.d sequence of random variables with standard normal distribution $\mathcal{N}(0,1)$, and being independent of $(\mathcal{F}_t)$. Let $G = 1_{[1,2)}(W_T)$ and $G_n = G + \psi_n$. We consider the increasing sequence of filtrations

$$\mathcal{G}_t^n = \bigcap_{s>t} \sigma(G_1,\ldots,G_n) \lor \mathcal{F}_s,$$

and put $\mathcal{G}_t^\infty = \bigcap_{s>t} \bigvee_{n} \mathcal{G}_s^n$. Notice that $\mathcal{G}_0^\infty$ contains all the information of $G$, since due to the law of large numbers $\lim_n \frac{1}{n} (G_1 + \ldots + G_n) = G$, a.s. As a consequence, under $(\mathcal{G}_t)$ there is arbitrage and hence $u^\infty(x) = \infty$.

Under $(\mathcal{G}_t^n)$ however, there are no arbitrage opportunities, since one can construct an ELMM for $(\mathcal{G}_t^n)$. Let $\pi_t((G_1,\ldots,G_n) \in da,\omega)$ be a regular conditional distribution with respect to $\mathcal{F}_t$. Then for almost all $\omega$ we have $\pi_t((G_1,\ldots,G_n)) \in da,\omega) \sim P_{(G_1,\ldots,G_n)}(da)$, where $P_{(G_1,\ldots,G_n)}$ is the joint distribution of $(G_1,\ldots,G_n)$. It follows from results f. ex. in [8], [1] that there exists an equivalent measure under which the vector $(G_1,\ldots,G_n)$ is independent of $\mathcal{F}_T$, and hence there exists an ELMM for $S$ under the filtration $(\mathcal{G}_t^n)$.

The preceding shows that the (NA) condition holds under $(\mathcal{G}_t^n)$. Therefore, $u^n(x) = u^n(x)$, $n \geq 1$. Now $u^n(x) \leq u^\infty(x)$. For $U = \log$ it has been shown in [3] that $u^\infty(x)$ is equal to the mutual information between $G$ and $S$, which is given by $I(G,S) = p \log p + (1-p) \log(1-p)$, where $p = P(G = 1)$. Consequently, the sequence $u^n(x)$ is bounded by $I(G,S) < \infty$, which shows that $\lim_n u^n(x) \neq u^\infty(x)$.

Finally, observe that $\sup_{a>0} u^\infty(x) = u^\infty(x) = \infty$. Therefore, there exists an $a > 0$ and an $a$-admissible strategy $\theta$ such that $EU(x + (\theta \cdot S)_T) > I(G,S)$. Consequently, $u^\infty(x) > I(G,S) \geq \lim sup_n u^n(x)$, which shows that Theorem 3.5 is not true without the assumption that $a \in (0, x]$. 

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3.3 Why not arbitrary price processes?

We close this section with an example showing that the Theorems 3.2 and 3.5 and Corollaries 3.3 and 3.7 are not valid without some regularity assumption on $S$.

**Example 3.9.** Let $T > 1$ and $\phi$ a random variable with standard normal distribution $\mathcal{N}(0, 1)$. Suppose the price process is given by

$$S_t = \begin{cases} 1, & \text{if } 0 \leq t < 1, \\ 1 + \phi + \frac{1}{2}, & \text{if } 1 \leq t \leq T, \end{cases}$$

and let $(\mathcal{F}_t^S)$ be the completed filtration generated by $S$. Moreover let $(\varepsilon_n)$ be a sequence of independent normal random variables with mean zero and $\text{Var}(\varepsilon_n) = \frac{1}{n}$. Let $\xi_n = 1_{\{|\phi| \geq 1\}} + \varepsilon_n$ and

$$\mathcal{G}_t^n = \mathcal{F}_t \lor \sigma(\xi_1, \ldots, \xi_n), \quad 0 \leq t \leq T.$$

We claim that

$$u^n(x) = U(x)$$

for all $x > 0$. For this let $\theta$ be $(\mathcal{G}_t^n)$-predictable and $S$-integrable. If $\theta_1 \neq 0$ a.s., then the integral $(\theta \cdot S)_1$ is unbounded from below and hence $\theta$ is not admissible. Since the process $S$ is constant on the remaining part of the trading interval, we have $u^n_0(x) = U(x)$. A trader having access to $\mathcal{G}_\infty^n = \bigvee_{n \geq 1} \mathcal{G}_t^n$ knows whether the absolute value of $\phi$ is bigger or smaller than 1. Therefore he has access to non-trivial admissible trading strategies. As a consequence $u^\infty(x) > U(x)$, and hence

$$\lim_n u^n_0(x) \neq u^\infty_0(x) \text{ and } \lim_n u^n(x) \neq u^\infty(x).$$

Note that the price process $S$ satisfies the (NA) condition with respect to $(\mathcal{G}_t^n)$. Therefore also in Corollary 3.7 the assumption that $S$ is continuous can not be dropped.

4 Monotone convergence with random endowments

We aim at proving a continuity property of utility based prices of contingent claims. To this end we have to generalise the results of the previous section to the case where the terminal wealth of an investor is not only determined by his investment strategy but also by a random payment or endowment due at time $T$. 


Let \( B \) be a random variable (endowment). For any semimartingale filtration \((\mathcal{H}_t)\) we define
\[
u^H(x, B) = \sup \{ EU(x + (\theta \cdot S)_T + B) : \theta \in \mathcal{A}(\mathcal{H}) \} \]
We will analyse to which extent the maximal expected utility with random endowment \( B \) has again the monotone convergence property. For this let again \((\mathcal{G}_t^n)\) be an increasing sequence with limit \((\mathcal{G}_t^\infty)\) so that the assumptions from section 2 are satisfied. As before we will use the short notation \( u^n(x, B) = u^{\mathcal{G}_n}(x, B) \) and \( u^\infty(x, B) = u^{\mathcal{G}_\infty}(x, B) \).

4.1 Convergence in the case \( \text{dom}(U) = \mathbb{R} \)

We need an integrability condition on our random endowment. We use the following one:
\[
EU(B + a) > -\infty \text{ for all } a \in \mathbb{R}. \tag{4}
\]
Note that if \( U \) is the exponential utility function, the integrability of \( U(B + a_0) \) for one \( a_0 \in \mathbb{R} \) implies already condition (4).

**Theorem 4.1.** Suppose (4). Then \( \lim_n u^n(x, B) = u^\infty(x, B) \).

**Proof.** Let \( \theta \) be \((\mathcal{G}_t^\infty)\)-predictable and admissible, let’s say \( a \)-admissible. Note that (4) implies \( EU(x + (\theta \cdot S)_T + B) > -\infty \). We can find \((\mathcal{G}_t^n)\)-predictable strategies \( \theta^n \) such that \( (\theta^n \cdot S) \to (\theta \cdot S) \) uniformly on \([0, T] \) a.s. By stopping at \( \tau_n = \inf \{ t \geq 0 : |((\theta^n - \theta) \cdot S)_t| \geq \delta \} \), where \( \delta > 0 \), we obtain \( (a + \delta) \)-admissible strategies \( \pi^n = 1_{[0, \tau_n]} \theta^n \) for which \( (\pi^n \cdot S)_T \to (\theta \cdot S)_T \), a.s. and
\[
x + (\pi^n \cdot S)_T + B \geq x - a - \delta + B.
\]
Since the negative part of \( U(x - a - \delta + B) \) is integrable, the negative parts of \( U(x + (\pi^n \cdot S)_T + B) \) are uniformly integrable. Therefore \( \lim_n E[U(x + (\pi^n \cdot S)_T + B)]^- = E[U(x + (\theta \cdot S)_T + B)]^- \). Consequently, by Fatou,
\[
\lim inf_n E[U(x + (\pi^n \cdot S)_T + B)] - E[U(x + (\theta \cdot S)_T + B)]^- = E[U(x + (\theta \cdot S)_T + B)]^- - E[U(x + (\theta \cdot S)_T + B)]^-\]
\[
\geq E[U(x + (\theta \cdot S)_T + B)]^- - E[U(x + (\theta \cdot S)_T + B)]^- = E[U(x + (\theta \cdot S)_T + B)]
\]
and hence \( \lim inf_n u^n(x, B) \geq u^\infty(x, B) \). Since obviously \( u^n(x, B) \leq u^\infty(x, B) \), the result.
4.2 Convergence in the case $\text{dom}(U) \neq \mathbb{R}$

Throughout we assume again that $\sup\{y : U(y) = -\infty\} = 0$. We start by showing that in this case a monotone convergence result as in Theorem 4.1 may no longer be true:

Example 4.2. Let $W$ be a Brownian motion with respect to a filtration $(\mathcal{F}_t)$. Let $dS_t = S_t(dW_t + adt)$ with $S_t > 0$, and let $B$ be independent of $(\mathcal{F}_t)$ and such that $P(B = -1) = \frac{1}{2} = P(B = 1)$, and let $(\mathcal{V}_t)$ be another Brownian motion which is independent of $B$ and $(\mathcal{F}_t)$. Let $\mathcal{H}_n = \sigma(B + V_r : r \geq \frac{1}{n})$ and $\mathcal{G}_n = \bigcap_{s \geq t} \mathcal{H}_n \vee \mathcal{F}_t$ for $n \geq 1$. The union $G_{\infty} = \bigcap_{s > t} \bigvee_n \mathcal{G}_n$ contains $\sigma(B)$. Now suppose that $U = \log$ and that $x > 1$. Under $(G_{t_n})$ we have

$$u_{n}(x, B) = u_{n+1}(x, B) \leq u_{\infty}(x, B) = \frac{1}{2}u_{x-1}^F(x - 1) + \frac{1}{2}u_{x+1}^F(x + 1).$$

Under $(G_{\infty})$ we know from the beginning whether $B = -1$ or $B = 1$. Therefore, $u_{\infty}(x, B) = \frac{1}{2}u_{x-1}^F(x - 1) + \frac{1}{2}u_{x+1}^F(x + 1)$. Note that $u_{x-1}^F(x + 1) \neq u_{x+1}^F(x + 1)$, and thus $\lim_n u_{n}(x, B) \neq u_{\infty}(x, B)$.

The example shows that we have to impose additional assumptions in order to guarantee convergence of $u_{n}(x, B)$ to $u_{\infty}(x, B)$. We continue with some auxiliary results that hold for any filtration $(\mathcal{F}_t)$ with respect to which $S$ is a semimartingale.

Lemma 4.3. Let $x > 0$, $a > 0$ and suppose that $EU(B + x) > -\infty$. Then

$$u_{a}^F(x, B) = \sup_{\varepsilon > 0} u_{a - \varepsilon}(x, B).$$

Proof. Let $\theta$ be an $a$-admissible strategy such that $EU(x + (\theta \cdot S)_T + B) > -\infty$. Put

$$\theta^n = (1 - \frac{1}{n})\theta$$

for all $n \geq 1$. Clearly $\theta^n$ is $(a - \frac{a}{n})$-admissible. Observe that $[U(x + (\theta^n \cdot S)_T + B) - U(x + B)]^- \leq [U(x + (\theta \cdot S)_T + B) - U(x + B)]^-$ and therefore $[U(x + (\theta^n \cdot S)_T + B) - U(x + B)]^-$ is uniformly integrable. By applying
Fatou’s lemma to the positive parts we obtain
\[
\liminf_n E[U(x + (\theta^n \cdot S)_T + B)] \\
= \liminf_n E[U(x + (\theta^n \cdot S)_T + B) - U(x + B)] + EU(x + B) \\
= EU(x + B) + \liminf_n E[U(x + (\theta^n \cdot S)_T + B) - U(x + B)]^+ \\
- \lim_n E[U(x + (\theta^n \cdot S)_T + B) - U(x + B)]^- \\
\geq EU(x + B) + E[U(x + (\theta \cdot S)_T + B) - U(x + B)]^+ \\
- E[U(x + (\theta \cdot S)_T + B) - U(x + B)]^- \\
= E[U(x + (\theta \cdot S)_T + B)].
\]

Therefore
\[
\liminf_n w_n^P(x, B) \leq \sup_{\varepsilon > 0} w_{n-\varepsilon}(x, B).
\]

Since the right hand side does obviously not exceed the left hand side, the proof is complete. \hfill \Box

Lemma 4.4. Let \(a, b > 0\). If \(\text{essinf}\, B \geq -b\), then for all \(x \geq a + b\)
\[
w_n^P(x, B) = \sup\{EU[x + (\theta \cdot S)_T + B] : \theta \in L^2_0(M) \cap A(\mathcal{F}), (a - \varepsilon) - \text{adm. for some } \varepsilon > 0\}. \tag{5}
\]

Proof. Let \(\theta \in A(\mathcal{F})\) be \((a - \varepsilon)\)-admissible. The stopping times \(T_n = T \wedge \inf\{t \geq 0 : \int_0^t \mathbf{1}_{[0,T_n]} \mathbf{1}_r d\langle M, M \rangle_r \leq n\}\) converge almost surely to \(T\). The strategies \(\theta^n = \mathbf{1}_{[0,T_n]} \theta\) are \((a - \varepsilon)\)-admissible and belong to \(L^2_0(M)\). Fatou’s Lemma implies \(\liminf_n EU[x + (\theta^n \cdot S)_T + B] \geq EU[x + (\theta \cdot S)_T + B]\), and thus, with Lemma 4.3, the result. \hfill \Box

Theorem 4.5. Let \(a, b > 0\). If \(\text{essinf}\, B \geq -b\), then for all \(x \geq a + b\)
\[
\lim_n w_n^P(x, B) = w_n^\infty(x, B).
\]

Proof. Let \(\theta \in L^2_0(M) \cap A(\mathcal{G})\) be \((a - \varepsilon)\)-admissible. By Lemma 4.4 it is enough to show that \(\lim_n u_n^P(x, B) \geq EU[x + (\theta \cdot S)_T + B]\). As in the proof to Theorem 3.2 we can construct \((a - \frac{\varepsilon}{2})\)-admissible and \((\mathcal{G}_t^\infty)\)-predictable strategies \(\pi^n\) so that \((\pi^n \cdot S)\) converges to \((\theta \cdot S)\) a.s. Fatou’s lemma implies
\[
\liminf EU[x + (\pi^n \cdot S)_T + B] \geq EU[x + (\theta \cdot S)_T + B],
\]
and hence the result. \hfill \Box

If \(B\) is independent of the filtration \((\mathcal{G}_t^\infty)\), then we can loosen the assumptions of the previous theorem.
Lemma 4.3 we obtain

\[ u \exists \text{a unique real number } \pi \]

Example 5.2. Let 
\[
\text{decrease} \text{ it will }
\]

Since \( u \) is attained by strategies satisfying (5), Convergence of indifference prices

\[
\text{essinf } EU \theta \text{ satisfies }
\]

\[
\text{positive probability and hence } EU[x + (\theta \cdot S)] + B = -\infty \text{. Therefore } u(x, B) \text{ is attained by strategies satisfying } (\theta \cdot S) \geq -x + b, \text{ a.s. Since we have (NA), } 
\]

\[ \theta \text{ must be } (x - b)\text{-admissible. As a consequence, } u(x, B) = u_{x - b}(x, B). \]

With Lemma 4.3 we obtain \( u(x, B) = \sup_{\varepsilon > 0} u_{x - b + \varepsilon}(x, B) \). Now we can proceed as in the proof of Theorem 4.5.

\[ \square \]

Note that Example 4.2 shows that the independence assumption can \textit{not} be skipped in the previous theorem.

5 Convergence of indifference prices

**Definition 5.1.** Let \( B \) be a random endowment and suppose that there exists a unique real number \( \pi \) such that \( u^H(x - \pi, B) = u^H(x) \). Then \( \pi \) is called the \textit{indifference price} or \textit{utility based price} of \( B \) relative to \( (H_t) \).

Do the indifference prices \( \pi_n \) of an increasing sequence of filtrations converge to the indifference price \( \pi_\infty \) of their union? Before we provide sufficient conditions for the convergence notice that if we enlarge the filtration, then a priori we do not know whether the indifference price will \textit{increase} or whether it will \textit{decrease}. This follows from the next example.

**Example 5.2.** Let \( U \) be a power utility function, i.e. \( U(x) = x^p \) for \( x \geq 0 \) and \( U(x) = -\infty \) if \( x < 0 \), where \( 0 < p < 1 \). Let \( S_t = W_t + \alpha t \), where \( \alpha > 0 \) and \( W \) is a Brownian motion with respect to \( (G_t) \).

Consider at first an endowment \( B \) which is independent of \( (G_t) \) and satisfies \( P(B = 0) = \frac{1}{2} = P(B = -1) \). The indifference price \( \pi_G \) of \( B \) under \( (G_t) \) has to satisfy \( u^G_B(1, B - \pi_G) = u^G(1) \). Observe that \( u^G_B(1, B - \pi_G) = u^G_B(1, B - \pi_G) \leq \frac{1}{2} u^G_B(-\pi_G) + \frac{1}{2} u^G_B(1 - \pi_G) \).

Now let \( H_t = \bigcap_{s \geq t} \sigma(B) \cup G_s \). If \( \pi_H \) is the indifference price under \( (H_t) \), then \( u^H_B(1, B - \pi_H) = \frac{1}{2} u^G_B(-\pi_H) + \frac{1}{2} u^G_B(1 - \pi_H) \). Note that \( u^H_B(1) = u^G_B(1) \), and therefore

\[
\frac{1}{2} u^G_B(-\pi_H) + \frac{1}{2} u^G_B(1 - \pi_H) \leq \frac{1}{2} u^G_B(-\pi_G) + \frac{1}{2} u^G_B(1 - \pi_G). \quad (6)
\]

Since \( \pi_G \) and \( \pi_H \) are negative, inequality (6) can only be satisfied if \( \pi_G < \pi_H \).
Now let \((\tilde{H}_t)\) be an enlargement of \((G_t)\) so that \(u^G(x) < u^\tilde{H}(x)\) for all \(x > 0\) and for which there exists an optimal strategy \(\theta^*\) with respect to \((\tilde{H}_t)\). Moreover suppose that \(S\) satisfies the (NA) condition with respect to \((\tilde{H}_t)\). As an example consider \(\tilde{H}_t = \bigcap_{s > t} \sigma(S_T + \psi) \vee G_s\), where \(\psi\) is independent of \(G_T\) and has the standard normal distribution.

Now let \(\tilde{B} = (\theta^* \cdot S)_T\) be the optimal pay-off under \((\tilde{H}_t)\). We claim that

\[
u^\tilde{H}(x, \tilde{B}) = u^\tilde{H}(x) = u^G(x, \tilde{B}). \tag{7}\]

Proof hereof: Suppose that \(\eta \in A(\tilde{H})\) is a strategy such that \(EU(x + (\eta \cdot S)_T + B) = EU[x + ((\eta + \theta^*) \cdot S)_T] > -\infty\). Note that \(S\) does not allow arbitrage. Therefore, with Lemma 3.6, the strategy \((\eta + \theta^*)\) is \(x\)-admissible. Consequently, \(E[U(x + (\eta \cdot S)_T + B)] \leq u^\tilde{H}(x)\). Hence \(u^\tilde{H}(x) \leq u^G(x, B) \leq u^{\tilde{H}}(x, \tilde{B}) = x^{\tilde{H}}(x)\), and thus (7).

From (7) we deduce immediately that under \((\tilde{H}_t)\) the indifference price of \(\tilde{B}\) is equal to zero. However, \(u^\tilde{H}(x) < u^G(x, \tilde{B})\), and the indifference price of \(\tilde{B}\) with respect to \((G_t)\) has to be greater than zero.

For the rest let \((G^n_t)\) be again an increasing sequence of filtrations converging to \((G^\infty_t)\) such that the assumptions of section 2 are satisfied. Let \(B\) be a random endowment. We will always assume here that under \((G^n_\infty)\) and each \((G^n_t)\) the indifference prices of \(B\) are defined, and we denote them by \(\pi^\infty\) and \(\pi^n\) respectively. In our analysis of convergence we start again with the simpler case \(\text{dom}(U) = \mathbb{R}\).

### 5.1 Convergence in the case \(\text{dom}(U) = \mathbb{R}\)

We will need the assumption that

\[
u^\infty(x)\text{ and } u^\infty(x, B)\text{ are finite and strictly increasing on } \mathbb{R}. \tag{8}\]

Notice that (8) implies that the indifference prices \(\pi^n\) and \(\pi^\infty\) are defined.

**Lemma 5.3.** Suppose (8). Then the sequence of indifference prices \((\pi^n)\) is bounded from below and from above.

**Proof.** We show at first that \((\pi_n)\) is bounded from above. Suppose they are not. Then we can find a subsequence converging to \(\infty\), and to simplify notation assume \(\lim_n \pi_n = \infty\). Concavity implies \(\lim_{x \to -\infty} u^\infty(x, B) = -\infty\). Therefore,

\[-\infty < u^\infty(x) = \lim_n u^n(x) = \lim_n u^n(x - \pi_n, B) \leq \lim_n u^\infty(x - \pi_n, B) = -\infty,\]

which is a contradiction.
Now we show that \((\pi_n)\) is bounded from below. Suppose that there exists a subsequence converging to \(-\infty\), and again to simplify notation assume \(\lim_n \pi_n = -\infty\). Let \(k < \pi_\infty\). Then due to Theorem 4.1,
\[
\lim_n u_n(x) = \lim_n u_n(x - \pi_n, B) \geq \lim_n u_n(x - k, B)
\]
\[
= u_\infty(x - k, B) > u_\infty(x - \pi_\infty, B) = u_\infty(x),
\]
which is a contradiction. Consequently, the indifference prices \(\pi_n\) are all in a bounded interval. □

**Theorem 5.4.** Suppose (8). Then \(\lim_n \pi_n = \pi_\infty\).

**Proof.** By Lemma 5.3 the indifference prices \(\pi_n\) are bounded, let’s say by \(C > 0\). Suppose that \(p \in [-C, C]\) is a cluster point of \((\pi_n)\), and let \((\lambda_n) = (\pi(n))\) be a subsequence of \((\pi_n)\) converging to \(p\). The concave functions \(u_n(x, B)\) converge pointwise to \(u_\infty(x, B)\). As a consequence, they converge uniformly on \([-C, C]\) (see Rockafella [11]). Therefore, \(\lim_n u_l(n)(x - \lambda_n, B) = u_\infty(x - p, B)\). Note that also \(\lim_n u_l(n)(x - \lambda_n, B) = \lim_n u_l(n)(x) = u_\infty(x) = u_\infty(x - \pi, B)\), and hence \(u_\infty(x - p, B) = u_\infty(x - \pi, B)\). Since \(u_\infty(x, B)\) is strictly increasing in \(x\), we have \(p = \pi\). As this is the case for any cluster point \(p\) of \((\pi_n)\), the result. □

Here a sufficient condition for the assumption (8) to be satisfied:

**Lemma 5.5.** Let \(B\) be bounded. If \(u_\infty(x)\) is finite and strictly increasing on \(\mathbb{R}\), then so is \(u_\infty(x, B)\).

**Proof.** It is straightforward to show that \(u_\infty(x, B)\) is concave and finite. Now suppose that it is not strictly increasing. Then it has to be constant finally, i.e. there exist constants \(d\) and \(e\) such that \(u_\infty(x, B) = d\) for all \(x \geq e\). Now observe that \(u_\infty(x - 2C, B) \leq u_\infty(x) \leq u_\infty(x, B)\), and therefore \(u_\infty(x) = d\) for all \(x \geq e + 2C\), which is a contradiction. □

### 5.2 Convergence in the case \(\text{dom}(U) \neq \mathbb{R}\)

Let \(\sup \{y : U(y) = -\infty\} = 0\). In this section we fix \(a > 0\) and consider only \(a\)-admissible strategies. Let \(B\) be an endowment so that \(\text{essinf} B \geq -b\) where \(0 \leq b < \infty\). We assume that

\[
u_\infty^a(x)\text{ and } u_\infty^a(x, B)\text{ are finite and strictly increasing on their domains.}
\]

In the remainder, \(\pi_n\) and \(\pi_\infty\) will be the indifference prices determined by \(u_\infty^a(x, B - \pi_n) = u_\infty^a(x)\) and \(u_\infty^a(x, B - \pi_\infty) = u_\infty^a(x)\) respectively.

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Theorem 5.6. Suppose (9). If \( x \geq a + b \), then the indifference prices satisfy

\[
\lim_n \pi_n = \pi_{\infty}.
\]

Proof. Again we need to show at first that \((\pi_n)\) is bounded from below. This can be done exactly in the same way as in the proof to Lemma 5.3: Suppose \( \lim_n \pi_n = -\infty \), and let \( k < \min(0, \pi_{\infty}) \). Then by Theorem 4.5 we have

\[
u_{\infty}^a(x) = \lim_n \nu_n^a(x - \pi_n, B) \geq \lim_n \nu_n^a(x - k, B) = \nu_{\infty}^a(x - k, B) > \nu_{\infty}^a(x - \pi_{\infty}, B) = \nu_{\infty}^a(x),
\]
a contradiction.

Note that \( \nu_n^a(a + b, B) \geq U(a) > -\infty \), and similarly, \( \nu_{\infty}^a(a + b, B) > -\infty \). Consequently we have uniform convergence of the concave functions on the compact interval \([a + b, C]\) where \( C \) is the upper bound of \((\pi_n)\), and we deduce the result as in the proof to Theorem 5.4. \( \square \)

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References


