Calculus via regularizations in Banach spaces: path dependent calculus and Kolmogorov type equations.

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STOCHASTIC ANALYSIS, CONTROLLED DYNAMICAL SYSTEMS AND APPLICATIONS

Jena, March 9-13th 2015

In honour of Prof. Dr. Hans-Jürgen ENGELBERT

Covers joint work with

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Hans-Jürgen ENGELBERT

- A great probabilist: a pioneer on stochastic differential equations with singular drift.
Outline

1. About a robust representation problem for random variables.
2. Finite dimensional calculus via regularization.
4. Window processes.
5. Towards a robust Clark-Ocone type formula.
7. Path-dependent semilinear Kolmogorov equation.
Basic survey reference

A. Cosso, C. Di Girolami, F. Russo (2014)

Calculus via regularizations in Banach spaces and Kolmogorov-type path-dependent equations.

http://arxiv.org/abs/1411.8000
Some related references to our work


Available preprints:
http://uma.ensta.fr/~russo/
1 About a robust representation problem for random variables.

1.1 Window processes

Let $X$ be a continuous process with quadratic variation $[X]$. 
Definition 1  Let $T > 0$ and $X = (X_t)_{t \in [0, T]}$ be a real continuous process prolonged by continuity. Process $X(\cdot)$ defined by

$$X(\cdot) = \{X_t(u) := X_{t+u}; u \in [-T, 0]\}$$

will be called window process.
6. $X(\cdot)$ is a $C([-T, 0])$-valued stochastic process.
6. $C([-T, 0])$ is a typical non-reflexive Banach space.

1.2 The robust representation

Is there a reasonable rich class of functionals

$$H : C := C([-T, 0]) \longrightarrow \mathbb{R}$$

such that the r.v.

$$h := H(X_T(\cdot))$$

admits a representation of the type
\[ h = V_0 + \int_0^T Z_s d^- X_s, \]

and

1. \( V_0 \in \mathbb{R}, \)

2. \( Z \) adapted process with respect to the canonical filtration of \( X. \)

Possibly we look for explicit expressions of \( V_0 \) and \( Z. \)
Idea: Representation of $h = H(X_T(\cdot))$

The idea consists in finding functions $u, v : [0, T] \times C \rightarrow \mathbb{R}$ such that $h = H(X_T(\cdot))$ as

$$V_t = u(t, X_t(\cdot)), \ Z_t = v(t, X_t(\cdot)), \ t \in [0, T],$$

$$V_t = h - \int_t^T Z_s d^- X_s, \ t \in [0, T].$$

In particular

$$h = u(0, X_0(\cdot)) + \int_0^T v(s, X_s(\cdot))d^- X_s.$$
   \[ u, v \] are related to a deterministic purely analytical tool as a PDE, in order to keep separated \textit{probability} and \textit{analysis}.

   Previous procedure make Clark-Ocone formula robust with respect to the quadratic variation.
Natural extensions.

\[ V_t = h - \int_t^T Z_s d^-X_s + \int_t^T F(s, X_s(\cdot), Y_s, Z_s) d[X]_s, \quad t \in [0, T], \]

for some \( F : [0, T] \times C([-T, 0] \times \mathbb{R} \times \mathbb{R}). \)

\[ [X] = \int_0^t \sigma^2(s, X_s(\cdot)) ds, \quad \sigma : [0, T] \times \mathbb{R} \rightarrow \mathbb{R}. \]
2 Finite dimensional calculus via regularization

Definition 2 Let $X$ (resp. $Y$) be a continuous (resp. locally integrable) process. Suppose that the random variables

$$
\int_0^t Y_s d^- X_s := \lim_{\epsilon \to 0} \int_0^t Y_s \frac{X_{s+\epsilon} - X_s}{\epsilon} dS
$$

exists in probability for every $t \in [0, T]$. 
If the limiting random function admits a continuous modification, it is denoted by \( \int_0^\cdot Y \, d^- X \) and called (proper) **forward integral** of \( Y \) with respect to \( X \). (FR-Vallois 1991)

If \( \lim_{t \to T} \int_0^t Y \, d^- X \) exists in probability we call that limit improper forward integral of \( Y \) with respect to \( X \), again denoted by \( \int_0^T Y \, d^- X \).
Covariation of real valued processes

Definition 3  The covariation of $X$ and $Y$ is defined by

$$[X, Y]_t = \lim_{\epsilon \to 0^+} \frac{1}{\epsilon} \int_0^t (X_{s+\epsilon} - X_s)(Y_{s+\epsilon} - Y_s)ds$$

if the limit exists in the ucp sense with respect to $t$. Obviously $[X, Y] = [Y, X]$. If $X = Y$, $X$ is said to be finite quadratic variation process and $[X] := [X, X]$. 
Connections with semimartingales

Let $S^1$, $S^2$ be $(\mathcal{F}_t)$-semimartingales with decomposition $S^i = M^i + V^i$, $i = 1, 2$ where $M^i$ $(\mathcal{F}_t)$-local continuous martingale and $V^i$ continuous bounded variation processes. Then

1. $[S^i]$ classical bracket and $[S^i] = \langle M^i \rangle$.  
2. $[S^1, S^2]$ classical bracket and $[S^1, S^2] = \langle M^1, M^2 \rangle$.  
3. If $S$ semimartingale and $Y$ cadlag and predictable

\[
\int_0^\cdot Y d^- S = \int_0^\cdot Y dS  \quad \text{(Itô)}
\]
Itô formula for finite quadratic variation processes

**Theorem 4** Let $F : [0, T] \times \mathbb{R} \rightarrow \mathbb{R}$ such that $F \in C^{1,2}([0, T] \times \mathbb{R})$ and $X$ be a finite quadratic variation process. Then

$$
\int_0^t \partial_x F(s, X_s) d^- X_s
$$

exists and equals

$$
F(t, X_t) - F(0, X_0) - \int_0^t \partial_s F(s, X_s) ds - \frac{1}{2} \int_0^t \partial_{xx} F(s, X_s) d[X]_s
$$
3 Stochastic calculus via
regularization in Banach spaces

A stochastic integral for $B^*$-valued integrand with
respect to $B$-valued integrators, which are not
necessarily semimartingales.

$\chi$-quadratic variation of $X$
A new concept of quadratic variation which generalizes
the tensor quadratic variation and which involves a
Banach subspace $\chi$ of $(B \hat{\otimes}_\pi B)^*$. 
Definition 5  Let $X$ (resp. $Y$) be a $B$-valued (resp. a $B^*$-valued) continuous stochastic process. Suppose that the random function defined for every fixed $t \in [0, T]$ by

$$
\int_0^t B^* \langle Y_s, d^-X_s \rangle_B := \lim_{\epsilon \to 0} \int_0^t B^* \langle Y_s, \frac{X_{s+\epsilon} - X_s}{\epsilon} \rangle_B d\epsilon
$$

in probability exists and admits a continuous version. Then, the corresponding process will be called \textit{forward stochastic integral of $Y$ with respect to $X$}.
Connection with Da Prato-Zabczyk integral

Let $B = H$ is separable Hilbert space.

**Theorem 6** Let $\mathbb{W}$ be a $H$-valued $Q$-Brownian motion with $Q \in L^1(H)$ and $\mathbb{Y}$ be $H^*$-valued process such that

$$\int_0^t \| \mathbb{Y}_s \|^2_{H^*} ds < \infty \text{ a.s.}$$

Then, for every $t \in [0, T]$, 

$$\int_0^t H^* \langle \mathbb{Y}_s, d\mathbb{W} \rangle_H = \int_0^t \mathbb{Y}_s \cdot d\mathbb{W}_s^{dz}$$

(Da Prato-Zabczyk integral)
Notion of Chi-subspace

Definition 7  A Banach subspace $\chi$ continuously injected into $(B \hat{\otimes}_\pi B)^*$ will be called Chi-subspace (of $(B \hat{\otimes}_\pi B)^*$).
In particular it holds

$$\| \cdot \|_{\chi} \geq \| \cdot \|_{(B \hat{\otimes}_\pi B)^*}.$$
Chi-quadratic variation

Let

1. $X$ be a $B$-valued continuous process,
2. $\chi$ a Chi-subspace of $\left(B \hat{\otimes}_\pi B\right)^*$,
3. $C([0, T])$ space of real continuous processes equipped with the ucp topology.

Two processes:

1. $[X] : \chi \to C([0, T])$;
2. $\widehat{[X]} : [0, T] \times \Omega \to \chi^*$ with bounded variation.
They are (loosely speaking) approached by $[\mathbb{X}]^\varepsilon$ be the applications

$$[\mathbb{X}]^\varepsilon : \chi \longrightarrow C([0, T])$$

defined by

$$\phi \mapsto \left( \int_0^t \chi \langle \phi, \frac{(\mathbb{X}_{s+\varepsilon} - \mathbb{X}_s) \otimes^2}{\varepsilon} \rangle_{\chi^*} \, ds \right)_{t \in [0, T]},$$
Definition 8  We say that $X$ admits a **global quadratic variation (g.q.v.)** if it admits a $\chi$-quadratic variation with

$$\chi = (B \hat{\otimes}_\pi B)^*.$$
When $\chi = (B \hat{\otimes}_\pi B)^*$

- H2 is related to weak* convergence in $(B \hat{\otimes}_\pi B)^{**}$. If $\Omega$ were a singleton then (H2) would imply

$$[\widetilde{X}]_t^\varepsilon(\Phi) \xrightarrow{\varepsilon \rightarrow 0} [\widetilde{X}]_t(\Phi), \quad \forall \Phi \in (B \hat{\otimes}_\pi B)^*, \quad t \in [0, T].$$

- The g.q.v. $[\widetilde{X}]$ is $(B \hat{\otimes}_\pi B)^{**}$-valued.
Infinite dimensional Itô’s formula

Let $B$ a separable Banach space

Theorem 9 Let $X$ a $B$-valued continuous process admitting a $\chi$-quadratic variation. Let $F : [0, T] \times B \rightarrow \mathbb{R}$ be $C^{1,2}$ Fréchet such that

$$D^2 F : [0, T] \times B \rightarrow \chi \subset (B \hat{\otimes}_\pi B)^*$$ continuously

Then for every $t \in [0, T]$ the forward integral

$$\int_0^t B^* \langle DF(s, X_s), d^- X_s \rangle_B$$

exists and the following formula holds.
\[ F(t, X_t) = F(0, X_0) + \int_0^t \partial_s F(s, X_s) ds + \]
\[ + \int_0^t B^* \langle DF(s, X_s), d^\pi X_s \rangle_B + \]
\[ + \frac{1}{2} \int_0^t \chi \langle D^2 F(s, X_s), d^{\widetilde{X}}_s \rangle_{\chi^*} \]
4 Window processes.

6 We fix now the attention on \( B = C = C([-T, 0]) \)-valued window processes.

6 \( X \) continuous real valued process and \( X(\cdot) \) its window process.

6 \( \overline{X} = X(\cdot) \)
If $X$ has Hölder continuous paths of parameter $\gamma > 1/2$, then $X(\cdot)$ has a zero g.q.v.

For instance:

- $X = B^H$ fractional Brownian motion with parameter $H > 1/2$.
- $X = B^{H,K}$ bifractional Brownian motion with parameters $H \in ]0, 1[,$ $K \in ]0, 1]$ s.t. $HK > 1/2$.

$W(\cdot)$ does not admit a global quadratic variation.
4.1 About a significant Chi-subspace

We shall consider the following Chi-subspace of $(C([-T, 0]) \hat{\otimes}_\pi C([-T, 0]))^*$:

$$\chi_0 := (\mathcal{D}_0 \oplus L^2([-T, 0]) \hat{\otimes}_h^2 \oplus \text{Diag},$$

with

$$\mathcal{D}_0 := \{ \lambda \, \delta_0(dx), \lambda \in \mathbb{R} \},$$

$$\text{Diag} := \{ \mu(dx, dy) = g(x) \delta_y(dx)dy; g \in L^\infty([-T, 0]) \}.$$
Evaluations of $\chi$-quadratic variation for window processes

Let $X$ be a real finite quadratic variation process.
$X(\cdot)$ has a $\chi_0$-quadratic variation and

$$[X(\cdot)]: \chi_0 \longrightarrow C[0,T]$$

$$[X(\cdot)]_t(\mu) = \int_{D_{-t}} d\mu(x,y)[X]_{t+x},$$

where $D_{-t} = \{(x,y)| - t \leq x = y = 0\}$. 
In particular, if $\mu \in (\mathcal{D}_0 \oplus L^2([-T, 0])) \hat{\otimes}_h^2$,

$$[X(\cdot)]_t(\mu) = \mu(\{0, 0\})[X]_t,$$
The particular case of a finite quadratic variation process $X$ such that $[X]_t = \int_0^t \sigma^2(s, X_s(\cdot))ds$.

If $\mu \in \chi_0$ then

$$[X, X]_t(\mu) = \int_{D_{-t}} d\mu(x, y) \int_0^{t+x} \sigma^2(s, X_s(\cdot))ds.$$ 

If $\mu \in (D_0 \oplus L^2([-T, 0])) \hat{\otimes}_h^2$, then

$$[X, X]_t = \mu(\{(0, 0)\}) \int_0^t \sigma^2(s, X_s(\cdot))ds.$$
Itô’s formula for the corresponding window process.

Let $F : [0, T] \times C \to \mathbb{R}$ of class $C^{1,2}$ such that $D^2 F : [0, T] \times C \to \chi_0$ continuous. We denote

$$DF(t, \eta) = D\delta_0 F(t, \eta)\delta_0(dx) + D^\perp F(t, \eta),$$

with $D\delta_0 F : [0, T] \times C \to \mathbb{R}$. The application of Itô formula gives

$$F(t, X_t) = F(0, X_0) + \int_0^t D\delta_0 F(s, X_s) d^- X_s + \int_0^t \mathcal{L}_s F(s, X_s) ds$$

$$+ \int_0^t \partial_s F(s, X_s) ds,$$

where
\[ \mathcal{L}_t G(\eta) = I(G)(t, \eta) + \frac{1}{2} \int_{D_{-t}} D^2_{dxdy} G(\eta) \sigma^2(t + x, \eta(x)), \]

\[ I(G)(t, \eta) = \int_{[-t,0]} D_{dx} G(\eta) d^- \eta(x), \]

provided that

\[ I(G)(t, \eta) = \lim_{\varepsilon \to 0} \int_{[-t,0]} D_{dx} G(\eta) \frac{\eta(x + \varepsilon) - \eta(x)}{\varepsilon}. \]

exists and \( I \) fulfills some technical assumptions.
5 Towards a Robust Clark-Ocone type formula

We set \( B = C = C([-T, 0]) \).

6 \( X \) real continuous stochastic process with values in \( \mathbb{R} \).
6 \( X_0 = 0 \),

6 \( [X]_t = \int_0^t \sigma^2(r, X_r(\cdot))dr \), where \( \sigma : [0, T] \times C \to \mathbb{R} \) is continuous.
Representation of $h = H(X_T(\cdot))$

Conformally to what we mentioned at the beginning, we aim at finding functions $u, v : [0, T] \times C \to \mathbb{R}$ such that

$$V_t = u(t, X_t(\cdot)), \quad Z_t = v(t, X_t(\cdot)),$$

$$V_t = h - \int_t^T Z_s d^- X_s, \quad t \in [0, T].$$

In particular

$$h = u(0, X_0(\cdot)) + \int_0^T v(s, X_s(\cdot)) d^- X_s.$$
5.1 A toy model (related to the Black-Scholes formula)

Let \((S_t)\) be the "price of a financial asset" of the type

\[ S_t = \exp(\sigma W_t - \frac{\sigma^2}{2} t), \quad \sigma > 0. \]

Let \(f : \mathbb{R} \to \mathbb{R}\) be a continuous function and

\[ h = \tilde{f}(S_T) = f(W_T) \text{ where } f(y) = \tilde{f} \left( \exp(\sigma y - \frac{\sigma^2}{2} T) \right). \]
Let \( \tilde{\mathcal{U}} : [0, T] \times \mathbb{R} \longrightarrow \mathbb{R} \) solving
\[
\begin{aligned}
\left\{ \quad & \partial_t \tilde{\mathcal{U}}(t, x) + \frac{\sigma^2}{2} \partial_{xx} \tilde{\mathcal{U}}(t, x) = 0 \\
& \tilde{\mathcal{U}}(T, x) = \tilde{f}(x) \quad x \in \mathbb{R}.
\end{aligned}
\]

Applying classical Itô formula we obtain
\[
\begin{align*}
h & = \tilde{\mathcal{U}}(0, S_0) + \int_0^T \partial_x \tilde{\mathcal{U}}(s, S_s) dS_s \\
& = \mathcal{U}(0, W_0) + \int_0^T \partial_x \mathcal{U}(s, W_s) dW_s,
\end{align*}
\]
for a suitable \( \mathcal{U} : [0, T] \times \mathbb{R} \longrightarrow \mathbb{R} \).
Robustness with respect to the volatility.

Does one have a similar representation if $W$ is replaced by a finite quadratic variation $X$ such that $[X]_t \equiv t$?

The answer is positive! because of Theorem 4.
Proposition 10  Let $X$ such that $[X]_t = \sigma^2 t$.

A1 $f : \mathbb{R} \rightarrow \mathbb{R}$ continuous and polynomial growth.

A2 $\mathcal{U} \in C^{1,2}([0, T] \times \mathbb{R}) \cap C^0([0, T] \times \mathbb{R})$ such that

$$\begin{cases} 
\partial_t \mathcal{U}(t, x) + \frac{\sigma^2}{2} \partial_{xx} \mathcal{U}(t, x) = 0 \\
v(T, x) = f(x). 
\end{cases}$$
Then

\[ h := f(X_T) = \mathcal{U}(0, X_0) + \int_0^T \partial_x \mathcal{U}(s, X_s) d^{-} X_s \]

improper forward integral

Natural question

Generalization to the case of "path dependent options"?
As first step we revisit the toy model.
5.2 The toy model revisited

Proposition 11  We set \( B = C = C([-T, 0]) \) and \( \eta \in C \) and we define

\[ H : C \to \mathbb{R}, \text{ by } H(\eta) := f(\eta(0)) \]

\[ u : [0, T] \times C \to \mathbb{R}, \text{ by } u(t, \eta) := \mathcal{U}(t, \eta(0)) \]

Then

\[ u \in C^{1,2} ([0, T] \times C; \mathbb{R}) \cap C^0 ([0, T] \times C; \mathbb{R}) \]

and solves
\[
\begin{aligned}
&\left\{ \begin{array}{l}
\partial_t u(t, \eta) + \frac{\sigma^2}{2} \int_{D_{-t}} D_{dx}^2 d_y u(t, \eta) = 0 \\
u(T, \eta) = H(\eta)
\end{array} \right.
\end{aligned}
\]

(Here $D_{\perp}u \equiv 0$).
Proof.

\[ u(T, \eta) = U(T, \eta(0)) = f(\eta(0)) = H(\eta) \]

\[ \partial_t U(t, \eta) = \partial_t U(t, \eta(0)) \]

\[ D u(t, \eta) = \partial_x U(t, \eta(0)) \delta_0 \]

\[ D^2 u(t, \eta) = \partial^2_{x x} U(t, \eta(0)) \delta_0 \otimes \delta_0 \]

\[ \partial_t u(t, \eta) + \frac{1}{2} D^2 u(t, \eta)(\{0, 0\}) = 0. \]
5.3 The general representation

The considerations of previous section bring us to the following.

**Theorem 12** Let us consider the following.

- $H : C \rightarrow \mathbb{R}$ continuous.
- $u \in C^{1,2} ([0, T] \times C) \cap C^0 ([0, T] \times C)$
- For any $(t, \eta) \in [0, T] \times C$, $\int_{[-t,0]} D_{dx}^+ u(t, \eta) \, d^- \eta(x)$ is well-defined (with a technical condition).
- $(t, \eta) \mapsto D^2 u(t, \eta)$ is continuous from $[0, T] \times C$ to $\chi_0$. 

Suppose that \(u\) solves the \(\text{Infinite dimensional PDE}\)

\[
\begin{aligned}
\begin{cases}
\partial_t u(t, \eta) + \mathcal{L}_t u(t, \eta) &= 0 \\
u(T, \eta) &= H(\eta).
\end{cases}
\end{aligned}
\]  

(1)
Then,

\[ h = V_0 + \int_0^T Z_s d^- X_s \]  

(in the possibly improper sense) with

- \( V_0 = u(0, X_0(\cdot)) \)
- \( Z_s = D^{\delta_0} u(s, X_s(\cdot)) \)
Definition 13  We call path dependent Kolmogorov PDE. related to $X$. 
6 Kolmogorov path dependent PDE.

The window of diffusion process

Let \( \sigma : [0, T] \times \mathbb{R} \rightarrow \mathbb{R} \) continuous and Lipschitz. Consider the stochastic flow \((X_t^{s,\xi})_{s \leq t \leq T}, \xi \in \mathbb{R}\), for \( s \in [t, T]\), \( \xi \in \mathbb{R} \), where, setting \( X = X^{s,\xi} \) solves the SDE

\[
X_t = \xi + \int_s^t \sigma(r, X_r) dW_r,
\]

where \((W, \mathcal{F}_t)\) is a classical Wiener process.
Remark 14 1. \( X \) is a.s. continuous in all the three variables \((s, t, \xi)\).

2. Let \( \sigma \) is of class \( C^{0,2}(\Delta \times \mathbb{R}) \), such that \( \sigma, \partial_x \sigma \) and \( \partial_{xx} \sigma \) are Hölder continuous, with

\[
\Delta = \{(s, t)|0 \leq s \leq t \leq T\}.
\]

Under previous assumptions we have, for \( k = 0, 1, 2 \),

\[
\sup_{0 \leq s \leq T} E\left( \sup_{t \in [s, T]} |\partial^{(k)}_{\xi} X^{s,\xi}_t|^p \right) \leq M,
\]

for every \( p \geq 1 \).
Associated with $X$, we link the following functional stochastic flow. For $\eta \in C$, we set

$$X_{s,\eta t}^{s,\eta}(x) = \begin{cases} \eta(t - s + x) : & x \leq s - t \\ X_{t+x}^{s,\eta(0)} : & x \geq s - t. \end{cases}$$

(3)

**Remark 15** If $\sigma = 1$ we denote the corresponding window Brownian flow by $W_{t}^{s,\eta} := X_{t}^{s,\eta}$.

From now on we will suppose $\sigma$ to be as in Remark 14.
Let \( h = H(X_T(\cdot)) \), where \( X = X_{T}^{0,x_0} \), for some \( x_0 \in \mathbb{R} \). We set

\[
V_t = E(h|\mathcal{F}_t).
\]

Then, there is \( u : [0, T] \times C \to \mathbb{R} \) such that

\[
V_t = u(t, X_t(\cdot)).
\]  

(4)

It is given by

\[
u(t, \eta) = E(H(X_T^{t,\eta})).\]
Aim.

Under suitable conditions on $H$, we aim to show that $u$ is the unique solution of the path dependent Kolmogorov equation (1), taking values in $C([ - T, 0])$.

This justifies to say that $u$ defined in (4) is the virtual solution for (1).
Definition 16  (Strict solution)

\[ u : [0, T] \times C \to \mathbb{R} \text{ of class } C^{1,2}([0, T] \times C) \cap C^{0}([0, T] \times C) \text{ is said to be a solution of (1) if} \]

1. \[ I(u)(t, \eta) := \int_{-t,0} D_{dx}u(t, \eta) d^+ \eta(x) \]
   is “well-defined”;

2. \( (t, \eta) \mapsto D^2u(t, \eta) \) is continuous into some Chi-subspace \( \chi \).

3. It solves effectively (1).
Sufficient conditions to find a strict solution of (1) and $\sigma = 1$.

Case $\sigma$ general in a paper in preparation (Cosso-Russo).

1. $H$ has a smooth Fréchet dependence on $C([-T, 0])$.

2. $h := H(X_T(\cdot)) = f \left( \int_0^T \varphi_1(s)d^-X_s, \ldots, \int_0^T \varphi_n(s)d^-X_s \right)$;

   $f : \mathbb{R}^n \rightarrow \mathbb{R}$ continuous and with linear growth;

   $\varphi_i \in C^2([0, T]; \mathbb{R}), \forall 1 \leq i \leq n$.

3. The determinant of $\Sigma_t := \left( \int_x^0 \varphi_i(y)\varphi_j(y)dy \right)$ is bigger than zero for every $x \in [-T, 0[$.
Uniqueness for solutions of the Kolmogorov type PDE.

Let \( u_1, u_2 : [0, T] \times C \to \mathbb{R} \) of class \( C^{1,2}([0, T] \times C) \cap C^0([0, T] \times C) \) being strict solutions of (1) of polynomial growth. If \( u_1, u_2 \) solve the Kolmogorov type equation then \( u_1 = u_2 \).
Sketch of the Proof.

We fix \((s, \eta) \in [0, T] \times C\). We have to prove that \(u_1(s, \eta) = u_2(s, \eta)\).

Without restriction of generality we set \(s = 0\).

1. We apply Itô formula to \(X_t = X^{0, \eta}_t\).
2. We take than the expectation and we obtain

\[
u_i(0, \eta) = E(H(X^0_T, \eta)).
\]

3. This shows the uniqueness.
7 Path-dependent semilinear Kolmogorov equation

Here we concentrate on the case $\sigma = 1$, the general case being *in Cosso-Russo (in preparation)*.

We study the path-dependent semilinear Kolmogorov equation:

\[
\begin{cases}
\partial_t U + \mathcal{L}t U = F(t, \eta, U, D^{\delta_0}U), & \forall (t, \eta) \in [0, T] \times C, \\
U(T, \eta) = G(\eta), & \forall \eta \in C,
\end{cases}
\]

where
\[ G : C([-T, 0]) \longrightarrow \mathbb{R} \]

\[ F : [0, T] \times C([-T, 0]) \times \mathbb{R} \times \mathbb{R} \longrightarrow \mathbb{R} \]

are Borel measurable functions. We refer to

\[ \partial_t U + L_i U, \]

as the \textit{path-dependent heat operator}. 
7.1 Strict solutions

**Definition 17** A function \( \mathcal{U} : [0, T] \times C([-T, 0]) \to \mathbb{R} \) in \( C^{1,2}([0, T] \times C) \cap C([0, T] \times C) \), which solves the path-dependent semilinear Kolmogorov equation, is called a **strict solution**.
Strict solutions: uniqueness

**Theorem 18 (Uniqueness)** Let $G : C([-T, 0]) \to \mathbb{R}$ and $F : [0, T] \times C([-T, 0]) \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ be Borel measurable functions satisfying, for some positive constants $C$ and $m$,

$$|F(t, \eta, y, z) - F(t, \eta, y', z')| \leq C(|y - y'| + |z - z'|),$$

$$|G(\eta)| + |F(t, \eta, 0, 0)| \leq C(1 + \|\eta\|^{m}_{\infty}),$$

for all $(t, \eta) \in [0, T] \times C([-T, 0]), y, y' \in \mathbb{R},$ and $z, z' \in \mathbb{R}$. 
Let $\mathcal{U} : [0, T] \times C([-T, 0]) \to \mathbb{R}$ be a strict solution to the path-dependent nonlinear Kolmogorov equation, satisfying the polynomial growth condition

$$|\mathcal{U}(t, \eta)| \leq C\left(1 + \|\eta\|_\infty^m\right), \quad \forall (t, \eta) \in [0, T] \times C([-T, 0]).$$
Then, we have

\[ U(t, \eta) = Y_{t, \eta}, \quad \forall (t, \eta) \in [0, T] \times C([-T, 0]), \]

where

\[
(Y_{s, \eta}^{t, \eta}, Z_{s, \eta}^{t, \eta})_{s \in [t, T]} = (U(s, W_{s, \eta}^{t, \eta}), D^{\delta_0} U(s, W_{s, \eta}^{t, \eta})1_{[t, T]}(s))_{s \in [t, T]}
\]

is the solution to the backward stochastic differential equation, \( P \)-a.s.,
\[ Y_{s}^{t,\eta} = G(W_{t}^{s}) + \int_{s}^{T} F(r, W_{r}^{t,\eta}, Y_{r}^{t,\eta}, Z_{r}^{t,\eta}) \, dr - \int_{s}^{T} Z_{r}^{t,\eta} \, dW_{r}, \]

for all \( t \leq s \leq T \). In particular, there exists at most one strict solution to the path-dependent nonlinear Kolmogorov equation.

Here \( W \) is the functional stochastic flow associated with \( W_{t}^{s,x} = x + (W_{t} - W_{s}) \) (classical Wiener process).
Alternative methods to strict solutions for Kolmogorov path dependent PDEs

- Dupire-Cont-Fournié (2010).
- Flandoli-Zanco (2014).
7.2 Strong-viscosity solutions: introduction

Various definitions of viscosity-type solutions for path-dependent PDEs have been given:

We propose a notion of solution which is not based on the standard definition of viscosity solution given in terms of test functions or jets.

### 7.3 Strong-viscosity solutions

**Idea and origin**

- Our notion of solution is defined, in a few words, as the pointwise limit of strict solutions to perturbed equations.
Our definition is more similar in spirit to the concept of *good solution*, which turned out to be equivalent to the definition of $L^p$-viscosity solution for certain fully nonlinear partial differential equations.

Our definition is likewise inspired by the notion of *strong solution*, even though strong solutions are required to be more regular than simply continuous.
Definition 19  A function $\mathcal{U} : [0, T] \times C([-T, 0]) \to \mathbb{R}$ is called a **strong-viscosity solution** to the path-dependent nonlinear Kolmogorov equation if there exists a sequence $(\mathcal{U}_n, G_n, F_n)_n$ satisfying:

(i) $\mathcal{U}_n : [0, T] \times C([-T, 0]) \to \mathbb{R}$, $G_n : C([-T, 0]) \to \mathbb{R}$, and $F_n : [0, T] \times C([-T, 0]) \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ are equicontinuous functions such that, for some positive constants $C$ and $m$, independent of $n$,

$$
|F_n(t, \eta, y, z) - F_n(t, \eta, y', z')| \leq C(|y - y'| + |z - z'|),
$$

$$
|\mathcal{U}_n(t, \eta)| + |G_n(\eta)| + |F_n(t, \eta, 0, 0)| \leq C(1 + \|\eta\|_\infty^m),
$$

for all $(t, \eta) \in [0, T] \times C([-T, 0])$, $y, y' \in \mathbb{R}$, and $z, z' \in \mathbb{R}$. 
(ii) $U_n$ is a strict solution to

$$
\begin{aligned}
\partial_t U_n + \mathcal{L}_t U_n &= F_n(t, \eta, U_n, D^{\delta_0} U_n), \\
U_n(T, \eta) &= G_n(\eta),
\end{aligned}
\forall (t, \eta) \in [0, T) \times C([-T, 0]), \\
\forall \eta \in C([-T, 0]).
$$

(iii) $$(U_n(t, \eta), G_n(\eta), F_n(t, \eta, y, z)) \rightarrow (U(t, \eta), G(\eta), F(t, \eta, y, z)), \text{ as } n \text{ tends to infinity, for any } (t, \eta, y, z) \in [0, T] \times C([-T, 0]) \times \mathbb{R} \times \mathbb{R}.$$
7.4 Strong-viscosity solutions: uniqueness

Theorem 20 (Uniqueness) Let \( \mathcal{U} : [0, T] \times C([-T, 0]) \rightarrow \mathbb{R} \) be a strong-viscosity solution to the path-dependent nonlinear Kolmogorov equation. Then, we have

\[
\mathcal{U}(t, \eta) = Y^{t, \eta}_t, \quad \forall (t, \eta) \in [0, T] \times C([-T, 0]),
\]

where \( (Y^{s, \eta}_s, Z^{s, \eta}_s)_{s \in [t, T]} \), with \( Y^{t, \eta}_s = \mathcal{U}(s, \mathbb{W}^{t, \eta}_s) \), solves the backward stochastic differential equation, \( P \)-a.s.,
\[ Y_{s}^{t,\eta} = G(W_{T}^{t,\eta}) + \int_{s}^{T} F(r, W_{r}^{t,\eta}, Y_{r}^{t,\eta}, Z_{r}^{t,\eta}) \, dr - \int_{s}^{T} Z_{r}^{t,\eta} \, dW_{r}, \]

for all \( t \leq s \leq T \). In particular, there exists at most one strong-viscosity solution to the path-dependent nonlinear Kolmogorov equation.
Strong-viscosity solutions: existence

Theorem 21 (Existence) Let $F \equiv 0$ and $G: C([-T, 0]) \to \mathbb{R}$ be uniformly continuous and satisfying the polynomial growth condition

$$|G(\eta)| \leq C(1 + \|\eta\|_\infty^m), \quad \forall \eta \in C([-T, 0]),$$

for some positive constants $C$ and $m$. Then, there exists a unique strong-viscosity solution $U$ to the path-dependent heat equation, which is given by

$$U(t, \eta) = \mathbb{E}[G(W^{t,\eta}_T)],$$

for all $(t, \eta) \in [0, T] \times C([-T, 0]).$
7.5 Paper in preparation.

A. Cosso, F. Russo.

Existence for functional dependent Kolmogorov type equation: the strict and strong-viscosity solution case.
Thank you for your attention.