AMERICAN OPTION VALUATION IN STOCHASTIC VOLATILITY MODELS WITH TRANSACTION COSTS

Andrea COSSO, Laboratoire de Probabilités et Modèles Aléatoires, Université Paris Diderot, France
Daniele MARAZZINA, POLITECNICO di MILANO
Carlo SGARRA, POLITECNICO DI MILANO

WORKSHOP ON STOCHASTIC ANALYSIS, CONTROLLED DYNAMICAL SYSTEMS AND APPLICATIONS

JENA – March 9 –13, 2015
PLAN OF THE TALK

- THE GENERAL FRAMEWORK
- THE FORMULATION AS A SINGULAR STOCHASTIC CONTROL PROBLEM
- EXISTENCE OF A VISCOSITY SOLUTION
- COMPARISON RESULTS AND UNIQUENESS
- NUMERICAL ILLUSTRATION
SOME REFERENCES


THE GENERAL FRAMEWORK

We suppose to have the following multidimensional stochastic process:

\[ dz(t) = rz(t)dt - (1 + \lambda)S(t)dL(t) + (1 - \mu)S(t)dM(t), \]  
\[ dy(t) = dL(t) - dM(t), \]  
\[ dS(t) = \alpha S(t)dt + \sqrt{\nu(t)}S(t)dW(t), \]  
\[ d\nu(t) = \xi(\eta - \nu(t))dt + \vartheta \sqrt{\nu(t)}dZ(t). \]
In the above equations $z$ represents the amount invested in the risk-free asset (the “Bond”), $S$ is the risky asset (the “Stock”), $r$ is the risk-free interest rate, $\alpha$ is the drift rate of the stock, $\lambda$ and $\mu$ are the (proportional) costs of buying and selling a stock, $\sqrt{\nu}$ is the volatility function, which we shall suppose to be driven by the Wiener process $Z$ according to a CIR-type dynamics.
The parameters $\xi$, $\eta$ and $\vartheta$ are assumed to be constant. To avoid a zero volatility we assume that $\xi\eta > \vartheta^2/2$ (the strict inequality is required in the proof of the comparison Theorem. The Wiener processes $W$, $Z$ are defined on a complete filtered probability space $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$, where the filtration $\mathbb{F} = (\mathcal{F}_t)_{0 \leq t \leq T}$, with $T > 0$ the final horizon, is the natural filtration generated by the two Wiener processes and satisfies the usual conditions.

$W$ and $Z$ are assumed to be correlated with coefficient $\rho$. $L(t)$ and $M(t)$ are the cumulative number of shares bought or sold, respectively, up to time $t \in [0, T]$. 
The cash value of a number of shares $y \in \mathbb{R}$ of the stock when its price is $S \in (0, +\infty)$ is not simply $yS$, but is given by

$$c(y, S) = \begin{cases} (1 + \lambda)yS, & \text{if } y < 0, \\ (1 - \mu)yS, & \text{if } y \geq 0, \end{cases}$$  \quad (5)$$

where $\lambda yS$ and $\mu yS$ are the amounts that the investor has to pay, due to the presence of the transaction costs.
We shall formulate the American option valuation problem as a utility maximization problem in strict analogy with the approach pioneered by Davis and Zariphopoulou.

Let $U : \mathbb{R} \to \mathbb{R}$ be the buyer utility function, assumed to be concave, increasing and such that $U(0) = 0$. Let us suppose that the buyer of the option owns an initial wealth $x$ in cash.

At time $t = 0$ he splits his wealth into the amounts $x_1$ and $x_2 = x - x_1$. He uses the quantity $x_1$ to buy $x_1/p$ shares of American options written on the risky asset $S$, where $p$ is the American option price we are going to define. The amount $x_2$, instead, is used to construct a portfolio $\pi$ composed by the bond and the stock, in order to maximize the expected utility of his terminal wealth:
\begin{align*}
V(t, z, y, S, \nu) &= \sup_{A_{t,T}} \mathbb{E}[U(z(T) + \\
&\quad + c(y(T), S(T))) | (z(t^-), y(t^-), S(t^-), \nu(t^-)) = (z, y, S, \nu)].
\end{align*}

Here $0 \leq t \leq T$, $(z, y, S, \nu) \in \mathbb{R} \times \mathbb{R} \times (0, +\infty) \times (0, +\infty)$ is the state at time $t^-$ and $A_{t,T}$ is the set of admissible trading strategies $(L, M)$ which we now define.
DEFINITION:

The set of admissible trading strategies $A_{t,T}$, for every $0 \leq t \leq T$, is the set of two-dimensional right-continuous, measurable, $\mathbb{F}$-adapted and increasing stochastic processes

$$(L, M) = (L(u), M(u))_{t \leq u \leq T}, L(t^-) = M(t^-) = 0.$$ 

Furthermore, $(L, M)$ are such that the corresponding processes $(z(u), y(u), S(u))_{t \leq u \leq T}$ satisfy:

$$(z(u), y(u), S(u)) \in S_{\tilde{K}}, \quad t \leq u \leq T,$$

where $\tilde{K}$ is a positive constant and

$$S_{\tilde{K}} = \{(z, y, S) \in \mathbb{R} \times \mathbb{R} \times (0, +\infty): z + c(y, S) > -\tilde{K}\}.$$
REMARK: Note that the set of admissible trading strategies \( \mathcal{A}_{t,T} \) depends also on the initial state \((z, y, S, \nu)\) at time \( t^- \).

The constraint is required in the proof of the comparison Theorem and it only rules out strategies which are clearly non-optimal, as the objective is the maximization of the utility of final wealth.

Moreover, we note that \( L(t) \) and \( M(t) \) may be positive, i.e., there can be a jump at the initial time \( t \).
At time $\tau$, the buyer could decide to exercise the option and to transfer the money to the portfolio, i.e., she receives the cash amount $Kx_1/p$ and pays to the option writer the amount $x_1S(\tau)/p$, which is the price of $x_1/p$ shares of the underlying security. If $[y(\tau), z(\tau)]$ is the investor’s portfolio composition at exercise time $\tau$, after the money transfers are performed the new portfolio composition is given by:

$$[y(\tau) - \frac{x_1}{p}, z(\tau) + \frac{Kx_1}{p}].$$

Therefore, let us define:

$$V_1(t, z, y, S, \nu; x_1) = V(t, z + \frac{Kx_1}{p}, y - \frac{x_1}{p}, S, \nu). \quad (9)$$
Then it is relevant to introduce the following value function:

\[ U(t,z,y,S,\nu; x_1) = \sup_{\mathcal{A}_{t,T,\tau}} \mathbb{E}[V_1(\tau, z(\tau), y(\tau), S(\tau), \nu(\tau); x_1) | (z, y, S, \nu)], \]

where \( \tau \in \mathcal{T}_{t,T} \), the set of \( \mathbb{F} \)-stopping times with values in \([t, T]\).

Now we define the auxiliary functions:

\[ \alpha(x_1, x_2, S, p, \nu) = U(0, x_2, 0, S, \nu; x_1), \quad (10) \]

\[ \beta(x, S, p, \nu) = \sup_{x_1 + x_2 = x} \alpha(x_1, x_2, S, p, \nu), \quad (11) \]

\[ X^*(p, S, x) = \arg \max \alpha(x_1, x - x_1, S, p, \nu). \quad (12) \]
We can finally define the writing price of the American option as follows:

\[ p^*(S) = \sup_x \{ p : X^*(p, S, x) > 0 \} \].

Hence, the fair price of the American option is defined to be the maximum price at which a positive investment is made in the option at time \( t = 0 \).

In the following sections, in order to avoid cumbersome notations, we shall drop the explicit dependence of \( V_1 \) on all the variables in some of the formulas presented.
THE SINGULAR CONTROL PROBLEM

We begin by restricting temporarily our interest to trading strategies which are absolutely continuous with respect to time, i.e., to those that can be written as:

\[ L(t) = \int_0^t \ell(s) \, ds, \quad M(t) = \int_0^t m(s) \, ds , \]

where \( \ell(s) \) and \( m(s) \) are nonnegative functions uniformly bounded by a fixed constant \( k < \infty \).
In this particular case, the previous equations become a vector stochastic differential equation with controlled drift and the value function of the approximate problem, denoted by $V_k$, satisfies the following HJB equation:

$$
\max_{0 \leq \ell, m \leq k} \left\{ \left( \frac{\partial V_k}{\partial y} - (1 + \lambda)S \frac{\partial V_k}{\partial z} \right) \ell - \left( \frac{\partial V_k}{\partial y} - (1 - \mu)S \frac{\partial V_k}{\partial z} \right) m \right\} + \\
+ \frac{\partial V_k}{\partial t} + \mathcal{L}V_k = 0,
$$
with terminal condition

$$V_k(T, z, y, S, \nu) = U(z + c(y, S))$$

for $$(z, y, S, \nu) \in \mathbb{R} \times \mathbb{R} \times (0, +\infty) \times (0, +\infty)$$.

Here the differential operator $L$ is given by:

$$LW = rz \frac{\partial W}{\partial z} + \alpha S \frac{\partial W}{\partial S} + \frac{1}{2} \nu S^2 \frac{\partial^2 W}{\partial S^2} + \xi (\eta - \nu) \frac{\partial W}{\partial \nu} + \frac{1}{2} \vartheta^2 \nu \frac{\partial^2 W}{\partial \nu^2} + \rho \vartheta \nu S \frac{\partial^2 W}{\partial S \partial \nu}.$$
The optimal trading strategy can be described by considering the following three possible cases:

• \( \frac{\partial V_k}{\partial y} \left(1 + \lambda\right) S \frac{\partial V_k}{\partial z} \geq 0 \),
  where the maximum is achieved by taking \( m = 0 \) and buying at the maximum possible rate \( \ell = k \);

• \( \frac{\partial V_k}{\partial y} \left(1 - \mu\right) S \frac{\partial V_k}{\partial z} \leq 0 \),
  where the maximum is achieved by taking \( \ell = 0 \) and selling at the maximum possible rate \( m = k \);
\[(1 - \mu)S \frac{\partial V_k}{\partial z} \leq \frac{\partial V_k}{\partial y} \leq (1 + \lambda)S \frac{\partial V_k}{\partial z},\]

where the maximum is achieved with \( \ell = 0 \) and \( m = 0 \), i.e., by neither buying nor selling.
These remarks suggest that the optimization problem turns out to be a free boundary problem, where, once the value function is known in the 5-dimensional space, defined by the state of the investor \((t, z, y, S, \nu)\), the optimal trading strategy is determined by the previous inequalities.

Moreover, the state space is divided into three regions called the Buy, Sell and No-Transaction regions, characterized by the same previous inequalities. In the limit \(k \to \infty\) the class of admissible trading strategies becomes the class defined before.
Then we conjecture that the state space remains divided into a Buy, a Sell and a No-Transaction region, where the value function satisfies the following set of equations:

- In the Buy region we have:

\[ V(s, z, y, S, \nu) = V(s, z - (1 + \lambda)S\delta y_b, y + \delta y_b, S, \nu), \]

where \( \delta y_b \), the number of shares bought by the investor, can take any positive value up to the number required to reach the boundary of the Buy region; when \( \delta y_b \to 0 \) the previous equation becomes:

\[ \frac{\partial V}{\partial y} - (1 + \lambda)S \frac{\partial V}{\partial z} = 0. \]
• In the Sell region the value function must satisfy the following equation:

\[ V(s, z, y, S, \nu) = V(s, z + (1 - \mu)S\delta y_s, y - \delta y_s, S, \nu), \]

where \( \delta y_s \), the number of shares sold by the investor, can take any positive value up to the number required to reach the boundary of the Sell region. In the limit \( \delta y_s \to 0 \) the previous equation becomes:

\[ \frac{\partial V}{\partial y} + (1 - \mu)S\frac{\partial V}{\partial z} = 0. \]

• In the No-Transaction region the value function is the solution of the following equation:
\[-\frac{\partial V}{\partial t} - \mathcal{L}V = 0 \quad \cdots \quad (13)\]

and the following inequalities must hold:

\[(1 - \mu)S\frac{\partial V}{\partial z} \leq \frac{\partial V}{\partial y} \leq (1 + \lambda)S\frac{\partial V}{\partial z}.\]
A direct inspection of the sign of the left hand side suggests that this is positive in both the Buy and the Sell regions, in such a way that the set of equations provided above can be condensed in the following fully non-linear PDE:

$$\min \left\{ -\frac{\partial V}{\partial y} + (1+\lambda)S \frac{\partial V}{\partial z}, \frac{\partial V}{\partial y} - (1-\mu)S \frac{\partial V}{\partial z}, -\frac{\partial V}{\partial t} - \mathcal{L}V \right\} = 0, \quad (14)$$

for \((s, z, y, S, \nu) \in [0, T) \times \mathbb{R} \times \mathbb{R} \times (0, +\infty) \times (0, +\infty)\).
With regard to $U$, we remark that it follows from the definition of $U$ that:

$$U(t, z, y, S, \nu) \geq V_1(t, z, y, S, \nu), \quad \text{on} [0, T) \times \overline{S_K} \times (0, +\infty).$$

Therefore, using for $U$ the same arguments just used for $V$ and taking into account the last inequality, we obtain, at least formally, the variational inequality that $U$ must satisfy:

$$\min \{U - V_1, -\frac{\partial U}{\partial y} + (1+\lambda)S\frac{\partial U}{\partial z}, \frac{\partial U}{\partial y} - (1-\mu)S\frac{\partial U}{\partial z}, -\frac{\partial U}{\partial t} - LU\} = 0.$$  

(15)
VISCOSITY PROPERTIES OF THE VALUE FUNCTIONS

In the present section we characterize the two value functions $V$ and $U$ as the unique constrained viscosity solutions to the corresponding HJB equations. To this end, we consider a general Hamilton-Jacobi-Bellman equation of the form:

$$F(t, X, W, \frac{\partial W}{\partial t}, D_X W, D_X^2 W) = 0, \text{ in } [0, T) \times S,$$

where $S$ is an open subset of $\mathbb{R}^n$, moreover $D_X W$ and $D_X^2 W$ are the gradient and the Hessian matrix of $W$ with respect to $X$, respectively. We write the state vector as $X = (X_1, X_2) \in S_1 \times S_2 = S$, where $X_1$ includes all the state variables on which some constraint is imposed, while $X_2$ is the set of state variables which is not subject to any constraints.
In our model the state $X$ corresponds to $(z, y, S, \nu)$, with $X_1 = (z, y, S)$ and $X_2 = \nu$.

Moreover, the set $S = S_1 \times S_2$ is given by $S_1 = \overline{S}_K$ and $S_2 = (0, +\infty)$.

We assume that the function $F: [0, T] \times \overline{S}_1 \times S_2 \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^{n \times n}$ is continuous and degenerate elliptic, i.e., for all $n \times n$ symmetric matrices $M, \hat{M}$ we have

$$F(t, X, r, q, p, M) \geq F(t, X, r, q, p, \hat{M}), \quad \text{if } M \leq \hat{M}. \quad (17)$$

Now we provide the definition of constrained viscosity solution to our problem.
DEFINITION: A continuous function $W: [0, T] \times \overline{S}_1 \times S_2 \rightarrow \mathbb{R}$ is a constrained viscosity solution of (18) on $[0, T) \times \overline{S}_1 \times S_2$ if:

- $W$ is a viscosity subsolution of (18) on $[0, T) \times \overline{S}_1 \times S_2$, that is for all $(t_0, X_0) \in [0, T) \times \overline{S}_1 \times S_2$ and for all $\varphi \in C^{1,2}([0, T) \times \overline{S}_1 \times S_2)$ such that $(t_0, X_0)$ is a local maximum point of $W - \varphi$ we have

$$F(t_0, X_0, W(t_0, X_0), \frac{\partial \varphi}{\partial t}(t_0, X_0), D_X \varphi(t_0, X_0), D^2_X \varphi(t_0, X_0)) \leq 0.$$ 

- $W$ is a viscosity supersolution of (18) on $[0, T) \times S_1 \times S_2$, that is for all $(t_0, X_0) \in [0, T) \times S_1 \times S_2$ and for all $\varphi$ \in
$C^{1,2}([0, T) \times \mathcal{S}_1 \times \mathcal{S}_2)$ such that $(t_0, X_0)$ is a local minimum point $W - \varphi$ we have

$$F(t_0, X_0, W(t_0, X_0), \frac{\partial \varphi}{\partial t}(t_0, X_0), D_X \varphi(t_0, X_0), D^2_X \varphi(t_0, X_0)) \geq 0.$$
THEOREM: The value function $U$ is a constrained viscosity solution of

$$\min \{ W-V_1, -\frac{\partial W}{\partial y} + (1+\lambda)S \frac{\partial W}{\partial z}, \frac{\partial W}{\partial y} - (1-\mu)S \frac{\partial W}{\partial z}, -\frac{\partial W}{\partial t} - LW \} = 0$$

(18)

on $[0,T) \times \bar{S}_K \times (0, +\infty)$. 
PROOF We separate the proof into two steps.

1) $U$ is a viscosity SUBSOLUTION.

Let $(t_0, X_0) \in [0, T) \times \overline{S}_K \times (0, +\infty)$, with $X_0 := (z_0, y_0, S_0, \nu_0)$, and $\varphi \in C^{1,2}([0, T) \times \overline{S}_K \times (0, +\infty))$ such that $(t_0, X_0)$ is a local maximum point of $U - \varphi$. Without loss of generality, we can assume that $U(t_0, X_0) = \varphi(t_0, X_0)$ and $U \leq \varphi$ on $[0, T) \times \overline{S}_K \times (0, +\infty)$.

We have to prove that

$$
\min \{ \varphi(t_0, X_0) - V_1(t_0, X_0), -\frac{\partial \varphi}{\partial y}(t_0, X_0) + (1+\lambda)S_0 \frac{\partial \varphi}{\partial z}(t_0, X_0), \frac{\partial \varphi}{\partial y}(t_0, X_0) - (1-\mu)S_0 \frac{\partial \varphi}{\partial z}(t_0, X_0), -\frac{\partial \varphi}{\partial t}(t_0, X_0) - \mathcal{L}\varphi(t_0, X_0) \} \leq 0.
$$

This amounts to say that at least one argument in the minimum operator must be non positive.
First we observe that $\varphi(t_0, X_0) \geq V_1(t_0, X_0)$, using the definition of $U$ and the equality $U(t_0, X_0) = \varphi(t_0, X_0)$. If $\varphi(t_0, X_0) = V_1(t_0, X_0)$ we get the thesis. Hence, we suppose that

$$\varphi(t_0, X_0) - V_1(t_0, X_0) > 0.$$ 

Now we argue by contradiction assuming that

$$\frac{\partial \varphi}{\partial y}(t_0, X_0) - (1 + \lambda)S_0 \frac{\partial \varphi}{\partial z}(t_0, X_0) < 0 \tag{19}$$

$$\frac{\partial \varphi}{\partial y}(t_0, X_0) - (1 - \mu)S_0 \frac{\partial \varphi}{\partial z}(t_0, X_0) > 0 \tag{20}$$

and

$$\frac{\partial \varphi}{\partial t}(t_0, X_0) + \mathcal{L}\varphi(t_0, X_0) < 0. \tag{21}$$
From the dynamic programming principle for $U$ we have

$$U(t_0, X_0) = \max \{ \sup_{\ell \in \mathbb{R}^+} U(t_0, z_0 - (1 + \lambda)S_0\ell, y_0 + \ell, S_0, \nu_0),$$

$$\sup_{m \in \mathbb{R}^+} U(t_0, z_0 + (1 - \mu)S_0m, y_0 - m, S_0, \nu_0) \}.$$  

Suppose that there exists $\bar{\ell} > 0$ such that

$$U(t_0, X_0) = \sup_{\ell \in [\bar{\ell}, +\infty)} U(t_0, z_0 - (1 + \lambda)S_0\ell, y_0 + \ell, S_0, \nu_0). \quad (22)$$
Then, using the dynamic programming principle, we deduce that

\[ U(t_0, X_0) = U(t_0, z_0 - (1 + \lambda)S_0 \ell, y_0 + \ell, S_0, \nu_0), \quad 0 \leq \ell \leq \bar{\ell}. \]

From this equality we obtain

\[ \varphi(t_0, z_0 - (1 + \lambda)S_0 \ell, y_0 + \ell, S_0, \nu_0) - \varphi(t_0, X_0) \geq 0, \quad 0 \leq \ell \leq \bar{\ell}. \]

As a consequence, dividing by \( \ell \) and taking the limit as \( \ell \) tends to 0, we get

\[ \frac{\partial \varphi}{\partial y}(t_0, X_0) - (1 + \lambda)S_0 \frac{\partial \varphi}{\partial z}(t_0, X_0) \geq 0, \]

which gives a contradiction. Then

\[ U(t_0, X_0) > U(t_0, z_0 - (1 + \lambda)S_0 \ell, y_0 + \ell, S_0, \nu_0), \quad \forall \ell > 0. \]

(23)
In an analogous way we can prove that
\[ U(t_0, X_0) > U(t_0, z_0 + (1 - \mu)S_0m, y_0 - m, S_0, \nu_0), \quad \forall m > 0. \]

(24)

Now, it remains to show that if (21) holds, too, then we get a contradiction. Note that using the continuity of \( V_1 \) and the smoothness of \( \varphi \) we deduce the existence of \( \delta > 0 \) such that
\[ \frac{\partial \varphi}{\partial y}(t, X) - (1 + \lambda)S\frac{\partial \varphi}{\partial z}(t, X) < 0 \]

and
\[ \frac{\partial \varphi}{\partial y}(t, X) - (1 - \mu)S\frac{\partial \varphi}{\partial z}(t, X) > 0, \]

for every \((t, X) \in \mathcal{B}(t_0, X_0) : = (t_0 - \delta, t_0 + \delta) \times B_\delta(X_0) \cap [0, T) \times \overline{S_K} \times (0, +\infty)\), where \( B_\delta(X_0) \) is the open ball of radius \( \delta \) centered at \( X_0 \).
Now, let $\varepsilon > 0$, then by using the dynamic programming principle we find two controls $L_\varepsilon$, $M_\varepsilon$ and a stopping time $\tau_\varepsilon$ such that for every stopping time $\tilde{\tau}$ we have:

$$U(t_0, X_0) \leq \mathbb{E}\left[U(\tilde{\tau}, X_{L_\varepsilon, M_\varepsilon}(\tilde{\tau}))1\{\tilde{\tau} \leq \tau_\varepsilon\} + V_1(\tau_\varepsilon, X_{L_\varepsilon, M_\varepsilon}(\tau_\varepsilon))1\{\tilde{\tau} > \tau_\varepsilon\}|X_{L_\varepsilon, M_\varepsilon}(t_0^-) = X_0\right] + \varepsilon,$$

where $X_{L_\varepsilon, M_\varepsilon}$ is the state process corresponding to the controls $L_\varepsilon$ and $M_\varepsilon$. 
Let introduce the following stopping time:

$$\bar{\tau} = \inf \{ t \in [t_0, T]: (t, X^{L_\varepsilon, M_\varepsilon}(t)) \notin B(t_0, X_0) \}.$$ 

We have that \( \mathbb{P}(\bar{\tau} > t_0) = 1 \).

Indeed, we can choose \( L_\varepsilon \) and \( M_\varepsilon \) with no jumps at time \( t_0 \). Since \( U(t_0, X_0) > V_1(t_0, X_0) \), we note that there exists an event \( A \in \mathcal{F}_{t_0} \), with \( \mathbb{P}(A) > 0 \), such that \( \tau_\varepsilon(\omega) > t_0 \) for every \( \omega \in A \).
Let $\tau$ be a stopping time such that $\tau \leq \bar{\tau} \wedge \tau_\varepsilon$. Applying Ito's formula to $\varphi(t, X^{L_\varepsilon, M_\varepsilon}(t))$ we get

$$
\mathbb{E}[\varphi(\tau, X^{L_\varepsilon, M_\varepsilon}(\tau))|X^{L_\varepsilon, M_\varepsilon}(t_0) = X_0] = \varphi(t_0, X_0) + \mathbb{E}[\int_{t_0}^{\tau} \left( \frac{\partial \varphi}{\partial t}(t, X^{L_\varepsilon, M_\varepsilon}(t)) + L\varphi(t, X^{L_\varepsilon, M_\varepsilon}(t)) \right) dt | X^{L_\varepsilon, M_\varepsilon}(t_0) = X_0]
$$

$$
- (1+\lambda)S^{L_\varepsilon, M_\varepsilon}(t) \frac{\partial \varphi}{\partial z}(t, X^{L_\varepsilon, M_\varepsilon}(t)) dL_\varepsilon(t) + \int_{t_0}^{\tau} ((1-\mu)S^{L_\varepsilon, M_\varepsilon}(t) \frac{\partial \varphi}{\partial z}(t, X^{L_\varepsilon, M_\varepsilon}(t)) - \frac{\partial \varphi}{\partial y}(t, X^{L_\varepsilon, M_\varepsilon}(t))) dM_\varepsilon(t)
$$

$$
\leq \varphi(t_0, X_0) + \mathbb{E}[\int_{t_0}^{\tau} \left( \frac{\partial \varphi}{\partial t}(t, X^{L_\varepsilon, M_\varepsilon}(t)) + L\varphi(t, X^{L_\varepsilon, M_\varepsilon}(t)) \right) dt | X^{L_\varepsilon, M_\varepsilon}(t_0) = X_0].
$$
By using the fact that $U \leq \varphi$, $U(t_0, X_0) = \varphi(t_0, X_0)$ and inequality before with $\tilde{\tau} = \tau$, we find

$$\varphi(t_0, X_0) - \varepsilon \leq \varphi(t_0, X_0) + \mathbb{E} \left[ \int_{t_0}^{\tilde{\tau}} \left( \frac{\partial \varphi}{\partial t} + \mathcal{L} \varphi \right)(t, X_{L \varepsilon}^{L \varepsilon, M \varepsilon}(t)) \, dt \mid X_{L \varepsilon, M \varepsilon}(t_0) = X_0 \right].$$

Hence

$$\mathbb{E} \left[ \int_{t_0}^{\tilde{\tau}} \left( \frac{\partial \varphi}{\partial t} + \mathcal{L} \varphi \right)(t, X_{L \varepsilon}^{L \varepsilon, M \varepsilon}(t)) \, dt \mid X_{L \varepsilon, M \varepsilon}(t_0) = X_0 \right] \geq -\varepsilon.$$
Let $\varepsilon' > 0$ and define the following stopping time:

$$
\tau' = \inf \{ t \in [t_0, T]: |(\frac{\partial \varphi}{\partial t} + \mathcal{L}\varphi)(t, X^{L_{\varepsilon},M_{\varepsilon}}(t)) - (\frac{\partial \varphi}{\partial t} + \mathcal{L}\varphi)(t_0, X_0)| > \varepsilon' \} \wedge \bar{\tau} \wedge \tau_{\varepsilon}.
$$

Then $\tau'(\omega) > t_0$ for every $\omega \in A$; therefore, by choosing $\varepsilon = \varepsilon' \mathbb{E}[^{\tau'} - t_0]$, we find

$$
(\frac{\partial \varphi}{\partial t} + \mathcal{L}\varphi)(t_0, X_0) \geq -2\varepsilon.
$$

From the arbitrariness of $\varepsilon'$ we find a contradiction.
2) $U$ is a viscosity SUPERSOLUTION.

Let $(t_0, X_0) \in [0, T) \times S_K \times (0, +\infty)$, with $X_0 := (z_0, y_0, S_0, \nu_0)$, and $\varphi \in C^{1,2}([0, T) \times S_K \times (0, +\infty))$ such that $(t_0, X_0)$ is a local minimum point of $U - \varphi$. We assume that $U(t_0, X_0) = \varphi(t_0, X_0)$ and $U \geq \varphi$ on $[0, T) \times S_K \times (0, +\infty)$. We have to prove that

$$\min \{ \varphi(t_0, X_0) - V_1(t_0, X_0), -\frac{\partial \varphi}{\partial y}(t_0, X_0) + (1+\lambda)S_0 \frac{\partial \varphi}{\partial z}(t_0, X_0), \frac{\partial \varphi}{\partial y}(t_0, X_0) - (1-\mu)S_0 \frac{\partial \varphi}{\partial z}(t_0, X_0), -\frac{\partial \varphi}{\partial t}(t_0, X_0) - \mathcal{L}\varphi(t_0, X_0) \} \geq 0.$$
Therefore we need to show that each argument of the minimum operator is nonnegative. Clearly $\varphi(t_0, X_0) \geq V_1(t_0, X_0)$, using the definition of $U$ and the fact that $U(t_0, X_0) = \varphi(t_0, X_0)$.

Now consider the trading strategy: $L(t) = \ell > 0$ and $M(t) = 0$, $t_0 \leq t \leq T$. By the dynamic programming principle

$$U(t_0, z_0, y_0, S_0, \nu_0) \geq U(t_0, z_0 - (1 + \lambda)S_0\ell, y_0 + \ell, S_0, \nu_0).$$

This inequality holds for $\varphi$ as well, and, by taking the left-hand side to the right-hand side, dividing by $\ell$, and sending $\ell \to 0$, we get

$$\frac{\partial \varphi}{\partial y}(t_0, X_0) - (1 + \lambda)S_0 \frac{\partial \varphi}{\partial z}(t_0, X_0) \leq 0.$$

In an analogous way we can prove that

$$\frac{\partial \varphi}{\partial y}(t_0, X_0) - (1 - \mu)S_0 \frac{\partial \varphi}{\partial z}(t_0, X_0) \geq 0.$$
Finally, to prove that the last argument inside the minimum operator is positive, consider the following trading strategy: \( L(t) = 0 \) and \( M(t) = 0 \), \( t_0 \leq t \leq T \). Denote by \( X^0(t) \) the corresponding dynamic of the state process. Thanks to the dynamic programming principle we have

\[
U(t_0, X_0) \geq \mathbb{E}[U(t, X^0(t))|X^0(t_0) = X_0], \quad t_0 \leq t \leq T.
\]

This inequality also holds for \( \varphi \). Applying Ito's formula to \( \varphi(t, X^0(t)) \) we get

\[
\mathbb{E}[\int_{t_0}^{t} (\frac{\partial \varphi}{\partial t} + L\varphi)(s, X^0(s))ds|X^0(t_0) = X_0] \leq 0.
\]

Therefore, dividing by \( t - t_0 \) and sending \( t \downarrow t_0 \) we deduce the thesis. This ends the proof.
We also have the following theorem regarding the value function $V$, whose proof is not reported, since is very similar to that of Theorem before.

**Theorem 1** The value function $V$ is a constrained viscosity solution of

$$\min \left\{ -\frac{\partial W}{\partial y} + (1 + \lambda)S\frac{\partial W}{\partial z}, \frac{\partial W}{\partial y} - (1 - \mu)S\frac{\partial W}{\partial z}, -\frac{\partial W}{\partial t} - \mathcal{L}W \right\} = 0$$

on $[0, T) \times \overline{S_K} \times (0, +\infty)$. 

(25)
In the present section we prove a comparison theorem, which allows us to show that the two value functions $V$ and $U$ are the unique constrained viscosity solutions to the corresponding HJB equations. We do this under the additional assumption, very useful also for numerical applications, that the utility function is of exponential type. More precisely, we assume that $U$ satisfies the following inequality for every $(z, y, S, \nu) \in \overline{S_K} \times (0, +\infty)$:

$$U(z + c(y, S)) \leq M - e^{-\gamma(z + c(y, S))},$$

(26)

where $M$ and $\gamma$ are positive constants.
THEOREM:

Suppose the previous inequality holds true. Let $u$ be a bounded upper semi-continuous viscosity subsolution of the problem on $[0, T) \times \overline{S_K} \times (0, +\infty)$ and $v$ be a lower semi-continuous function which is bounded from below, exhibits sublinear growth and is a viscosity supersolution on $[0, T) \times \overline{S_K} \times (0, +\infty)$.

Suppose that $u(T, X) \leq v(T, X)$ for every $X \in \overline{S_K} \times (0, +\infty)$. Then $u \leq v$ on $[0, T] \times \overline{S_K} \times (0, +\infty)$.

PROOF.

First we construct a positive strict supersolution on $[0, T] \times \overline{S_K} \times (0, +\infty)$ when $U$ satisfies the assumption above.

Let $\beta > 1/2$ be such that $\xi \eta > \beta \vartheta^2$. 

40
Then define $h: [0, T] \times \overline{S_K} \times (0, +\infty) \to \mathbb{R}$ as follows:

$$h(t, z, y, S, \nu) = M - e^{-\gamma(z+kyS)} + \frac{1}{\nu^{2\beta-1}} + C_1(T-t) + C_2,$$

where $C_1$ is a positive constant that will be fixed later, while the constant $k$ satisfies

$$1 - \mu < k < 1 + \lambda.$$

Finally, $C_2$ is a positive constant that makes $h$ strictly positive and $h \geq V_1 + K'$ on $[0, T] \times \overline{S_K} \times (0, +\infty)$, for some constant $K' > 0$ (we observe that $V_1$ is bounded on $[0, T] \times \overline{S_K} \times (0, +\infty)$, thanks to our assumption on $\mathcal{U}$). Note that $h(T, z, y, S, \nu) > \mathcal{U}(z + c(y, S))$, for every $(z, y, S, \nu) \in \overline{S_K} \times (0, +\infty)$, taking $C_2$ large enough. It remains to prove the strict supersolution property.
We have:

\[ H(t, X, h, \frac{\partial h}{\partial t}, D_X h, D_X^2 h) = \min \{ h - V_1, -\frac{\partial h}{\partial y} + (1 + \lambda)S\frac{\partial h}{\partial z}, -\frac{\partial h}{\partial y} - (1 - \mu)S\frac{\partial h}{\partial z}, -\frac{\partial h}{\partial t} - Lh \} \geq \]

\[ \geq e^{-\gamma(z + kyS)} \min \{ K'e^{\gamma(z + kyS)}, S(1 + \lambda - k), S(k - (1 - \mu)), C_1 e^{\gamma(z + kyS)} - r\gamma(z + kyS) + \frac{1}{2}\nu\gamma^2 k^2 y^2 S^2 - (\alpha - r)\gamma kyS + \xi(\eta - \nu)\frac{2\beta - 1}{\nu^{2\beta}} e^{\gamma(z + kyS)} - \nu^2 \frac{2(2\beta - 1)\beta}{\nu^{2\beta}} e^{\gamma(z + kyS)} \}. \]

Now we show that we can choose \( C_1 \) large enough in such a way that the last argument in the minimum operator is strictly positive. Note that the function \( D(\zeta) = \nu\gamma^2 k^2 \zeta^2 / 2 - (\alpha - r)\gamma ky\zeta \) has minimum value equal to \( -(\alpha - r)^2 / (2\nu) \).
Consequently, the last argument inside the minimum operator is greater than or equal to the following expression:

$$C_1 - \frac{(\alpha - r)^2}{2\nu} - \frac{\xi (2\beta - 1)}{\nu^{2\beta - 1}} + \frac{(\xi \eta - \beta \vartheta^2)(2\beta - 1)}{\nu^{2\beta}} e^{\gamma(z + k\gamma S)} - \nu \gamma(z + k\gamma S).$$

Since $\xi \eta > \beta \vartheta^2$, the function

$$G(\nu) = -\frac{(\alpha - r)^2}{(2\nu)} - \frac{\xi (2\beta - 1)}{\nu^{2\beta - 1}} + \frac{(\xi \eta - \beta \vartheta^2)(2\beta - 1)}{\nu^{2\beta}}$$

is bounded from below by a constant: $G(\nu) \geq -A$ for every $\nu > 0$

where $A$ is a positive constant. Take $C_1 = A + C_3$, where $C_3$ is a positive constant that will be fixed below.
Then

\[ C_1 - \frac{(\alpha-r)^2}{2 \nu} - \frac{\xi(2\beta-1)}{\nu^2 \beta - 1} + \left( \frac{\xi \eta - \beta \theta^2}{\nu^2 \beta} \right) \eta (2\beta - 1) e^{\gamma(z + kyS)} - r \gamma(z + kyS) \geq \]

\[ \geq C_3 e^{\gamma(z + kyS)} - r \gamma(z + kyS). \]

We can choose \( C_3 \) large enough so that the function \( F(x) = C_3 e^x - rx \), with \( x \geq -\gamma \bar{K} \), is bounded from below by a constant and, in particular, is strictly positive. In conclusion, we have proved that there exists a strictly positive constant \( \delta \) such that

\[ H(t, X, h, \frac{\partial h}{\partial t}, D_X h, D_X^2 h) \geq \delta, \quad (27) \]

on \([0, T) \times \bar{S} \bar{K} \times (0, +\infty)\).
To conclude the proof of the theorem, define the function $w^\varepsilon = (1 - \varepsilon)v + \varepsilon h$, with $0 < \varepsilon < 1$. Then $u(T, X) \leq w^\varepsilon(T, X)$ for every $X \in \overline{S}_K \times (0, +\infty)$. Moreover $w^\varepsilon$ is a viscosity supersolution of $H - \varepsilon \delta = 0$ on $[0, T) \times \overline{S}_K \times (0, +\infty)$. Now we may apply Lemma 4.1 in Davis and Zariphopoulou to $u$ and $w^\varepsilon$ and we deduce that $u \leq w^\varepsilon$ on $[0, T] \times \overline{S}_K \times (0, +\infty)$. Therefore, sending $\varepsilon \downarrow 0$, we get the thesis.

COROLLARY: Under the hypothesis made on the utility function, the value functions $U$ and $V$ are the unique constrained viscosity solutions to the respective HJB variational inequalities.

PROOF. Thanks to assumption made we have that both $U$ and $V$ are bounded. Therefore we may apply the comparison Theorem just proved and we deduce the thesis.
A NUMERICAL ILLUSTRATION

In the present section we assume a particular expression for the utility function describing the preferences of the investor. More precisely, we suppose that the utility function is a negative exponential utility of the following kind:

\[ U(x) = -\exp(-\gamma x). \]

Thanks to this assumption the dimensionality of the problem can be substantially reduced.

Moreover, the solution of the optimization problem does not depend on the investor’s initial wealth. This kind of utility function describes the preferences of an investor exhibiting constant risk-aversion, for this reason is sometimes called a CARA (Constant Absolute Risk Aversion) utility function.
This choice has been also adopted in both the papers by Davis and Zariphopoulou and by Zakamouline, dealing respectively with European and American option pricing with transaction costs with constant volatility.

This specific choice seems restrictive to some extent, but it has shown that the dependence of option prices on the specific form of the utility function is very weak. Therefore we decided to stick on this particular choice, which greatly simplifies the computational procedure.
Now we introduce the following discount factor:

$$\delta(t, T) = \exp(-r(T - t)),$$

and the “reduced utility functions”:

$$U(t, z, y, S, \nu) = \exp \left( -\gamma \frac{z}{\delta(t, T)} \right) Q(t, y, S, \nu),$$

$$V(t, z, y, S, \nu) = \exp \left( -\gamma \frac{z}{\delta(t, T)} \right) Q_0(t, y, S, \nu).$$

$$V_1(t, z, y, S, \nu) = \exp \left( -\gamma \frac{z}{\delta(t, T)} \right) Q_1(t, y, S, \nu).$$
REMARK. In order to slightly simplify the notation, and in analogy with the above mentioned papers, in the definition of $U$ and $V_1$ provided, we have assumed that a single option is purchased by the investor, in such a way that $x_1 = p$ and that, when the option exercise takes place, only the $z$ argument in the functions $U$ and $V_1$ changes by the amount $g(S) := \max(K - S, 0)$, while the $y$ argument is left unchanged.

According to this assumption, the definitions of $V_1$ and $U$ can be reformulated as follows:

$$V_1(t, z, y, S, \nu) = V(t, z + g(S), y, S, \nu),$$

which implies
\[ Q_1(t, z, y, S, \nu) = \exp \left( -\gamma \frac{g(S)}{\delta_{t,T}} \right) Q_0(t, y, S, \nu), \]

\[ U(t, z, y, S, \nu) = \]

\[ = \sup_{A_{t,T}, \tau} \mathbb{E}[V_1(\tau, z(\tau), y(\tau), S(\tau), \nu(\tau))| (z(t^-), y(t^-), S(t^-), \nu(t^-))]. \]

The purchase price of an American option simply becomes the value \( p \) such that

\[ V(t, z, y, S, \nu) = U(t, z - p, y, S, \nu), \]

i.e.,

\[ p = \frac{\delta(t, T)}{\gamma} \log \left( \frac{Q_0(t, y, S, \nu)}{Q(t, y, S, \nu)} \right). \]
As a consequence, we may express the variational inequalities for \( U \) and \( V \) in terms of \( Q, Q_0 \) and \( Q_1 \), obtaining:

\[
\min \{ Q - Q_1, -\frac{\partial Q}{\partial y} - \gamma (1 + \lambda) S \frac{Q}{\delta(t,T)}, \frac{\partial Q}{\partial y} + (1 - \mu) S \frac{Q}{\delta(t,T)}, -\frac{\partial Q}{\partial t} - \mathcal{D}W \} = 0,
\]

and

\[
\min \{ -\frac{\partial Q_0}{\partial y} - \gamma (1 + \lambda) S \frac{Q_0}{\delta(t,T)}, \frac{\partial Q_0}{\partial y} + (1 - \mu) S \frac{Q_0}{\delta(t,T)}, -\frac{\partial Q_0}{\partial t} - \mathcal{D}W \} = 0,
\]
where the operator $\mathcal{D}$ is defined as follows:

$$\mathcal{D}Q = \alpha S \frac{\partial Q}{\partial S} + \frac{1}{2} \nu S^2 \frac{\partial^2 Q}{\partial S^2} + \xi (\eta - \nu) \frac{\partial Q}{\partial \nu} + \frac{1}{2} \vartheta^2 \nu \frac{\partial^2 Q}{\partial \nu^2} + \rho \vartheta \nu S \frac{\partial^2 Q}{\partial S \partial \nu}.$$

Now, the functions $Q(t, y, S, \nu)$ and $Q_0(t, y, S, \nu)$ are defined on a 4-dimensional space $[0, T] \times \mathbb{R} \times (0, +\infty) \times (0, +\infty)$.

The terminal conditions are given by

$$Q_0(T, y, S, \nu) = -e^{-\gamma c(y, S)} \quad \text{and} \quad Q(T, y, S, \nu) = -e^{-\gamma (g(S) + c(y, S))}.$$
As in the papers by DPZ and Zakamouline, we can couple the variational HJB inequalities with a Markov chain approximation. More precisely, in both papers the authors deal with the classical lognormal model, solving the pricing problem with a binomial model. Moreover, in the second paper, the author provides an alternative characterization of the value function which is based on a global maximum, and that is well-suited for the application of the Markov chain approximation technique.

Since in this article we deal with stochastic volatility, first of all we have to introduce a tree-based method to price American options (without transaction costs) in the Heston model. Then we can easily couple this method with the variational HJB inequalities exploiting the Markov chain approximation.
We consider the tree-based model presented by Vellekoop and Nieuwenhuis: the pricing approach is based on a modification of a combined tree for stock prices and volatilities, where the value of the derivative is computed on a two-dimensional grid (in stock and volatility) at each time step, exploiting interpolating techniques. In all our experiments we deal with the bilinear interpolation technique suggested there. This pricing procedure prevents from the problem of dealing with non-recombining tree, which often happens when dealing with lattice methods for the Heston model.
Coupling the model presented with the Markov chain approximation, we obtain the following procedure to compute $Q$ and $Q_0$.

Let us consider a discrete time grid $\{0, \delta t, 2\delta t, \ldots, N\delta t\}$ with $N = T/\delta t$, $T$ being the maturity of the American derivative. The Markov chain for the discrete stock price $S(t)$ and volatility $\nu(t)$ processes are modeled according to VN, i.e.,

$$\nu((i + 1)\delta t) = \max \left( \nu(i\delta t) + \xi (\eta - \nu(i\delta t)) \delta t + Y^1 \vartheta \sqrt{\nu(i\delta t)\delta t}, 0 \right),$$

$$S((i + 1)\delta t) = S(i\delta t) e^{(\alpha - \frac{1}{2} \nu(i\delta t)) \delta t + Y^2 \sqrt{\nu(i\delta t)\delta t}},$$
Moreover, the discrete time equation for the amount invested in the risk-free asset is

\[ z((i + 1)\delta t) = z(i\delta t)e^{r\delta t}. \]

After defining a grid for the number of shares, i.e., \( y = y_j = j\delta y, j = -J, \cdots, J \), the discretization scheme for the HJB equation is the following:

\[ Q_0(i\delta t, y_j, S(i\delta t), \nu(i\delta t)) = \]

\[ = \max \left( \mathbb{E} \left[ Q_0((i + 1)\delta t, y_j, S((i + 1)\delta t), \nu((i + 1)\delta t)) \right], \right. \]

\[ F_b(i\delta t, \delta y, S(i\delta t))Q_0(i\delta t, y_{j+1}, S(i\delta t), \nu(i\delta t)), \]

\[ F_s(i\delta t, \delta y, S(i\delta t))Q_0(i\delta t, y_{j-1}, S(i\delta t), \nu(i\delta t)) \right). \]
with
\[ F_b(t, \delta y, S) = e^{\gamma \frac{(1+\lambda)\delta y S}{\delta(t,T)}} \]
and
\[ F_s(t, \delta y, S) = e^{-\gamma \frac{(1-\mu)\delta y S}{\delta(t,T)}} \; ; \]
and
\[ Q_0(i\delta t, y_{j\pm 1}, S(i\delta t), \nu(i\delta t)) = \]
\[ = \mathbb{E} \left[ Q_0((i + 1)\delta t, y_{j\pm 1}, S((i + 1)\delta t), \nu((i + 1)\delta t)) \right] , \]
where the expected values are computed exploiting the two-dimensional binomial tree.
Notice that the first line corresponds to do nothing, while the second (third) one to buy (sell) \( \delta y \) shares of the stock. Similarly, we have:

\[
Q(i\delta t, y_j, S(i\delta t), \nu(i\delta t)) =
\]

\[
= \max (\mathbb{E}[Q((i + 1)\delta t, y_j, S((i + 1)\delta t), \nu((i + 1)\delta t))],
F_b(i\delta t, \delta y, S(i\delta t))Q(i\delta t, y_{j+1}, S(i\delta t), \nu(i\delta t)),
F_s(i\delta t, \delta y, S(i\delta t))Q(i\delta t, y_{j-1}, S(i\delta t), \nu(i\delta t)),
Q_1(i\delta t, y_j, S(i\delta t), \nu(i\delta t))),
\]

where the last line corresponds to the early exercise of the American contract, and can be evaluated by \( Q_1 \) and \( Q_0 \).
The algorithm developed in the previous section was implemented, computing the price of an American put contract. In our numerical experiments we deal with the following parameters: \( r = \alpha = 0.1 \), and \( S(0) = 9 \), \( \nu(0) = 0.0625 \), \( \eta = 0.16 \), \( \vartheta = 0.9 \), \( \xi = 5 \), and \( \rho = 0.1 \).

Moreover, the American put has strike \( K = 10 \) and maturity \( T = 0.25 \). The discretization parameters of the Markov Chain are \( \delta t = 0.007 \), \( \delta y = 0.2 \), and \( J = 50 \).
In the previous Figure we plotted the American put price for different values of the parameter $\gamma$ and considering two different sets of proportional costs: $\lambda = \mu = 1\%$ and $\lambda = \mu = 0.01\%$, while in the following Table we deal with the influence of proportional transaction costs on the option price setting $\gamma = 0.1$.

<table>
<thead>
<tr>
<th>$\lambda = \mu$</th>
<th>0.0001</th>
<th>0.0005</th>
<th>0.001</th>
<th>0.005</th>
<th>0.01</th>
</tr>
</thead>
<tbody>
<tr>
<td>Price</td>
<td>1.1088</td>
<td>1.1033</td>
<td>1.0988</td>
<td>1.0848</td>
<td>1.0848</td>
</tr>
</tbody>
</table>
It is clear that the option price decreases when both the proportional transaction costs and the absolute risk-aversion $\gamma$ increase. These results are in line with what is presented in (Zakamouline 2005) when a classical lognormal model is considered. Therefore, as expected, moving from the lognormal model to the Heston stochastic volatility model does not change the behavior of the derivative price with respect to $\gamma$ and the proportional costs’ parameters $\lambda$ and $\mu$.

Moreover, decreasing $\lambda$, the option price approaches to its value when no transaction costs are considered.
Early exercise boundary at time $T/2$, setting $\lambda = \mu = 0.01$ (left) and $\gamma = 0.05$ (right).
To conclude, in the last Figure we see the early exercise boundary at time $\frac{T}{2}$ in the space $(S, \nu)$, i.e., underlying asset and variance, dealing with the same American put as above and considering different values of the risk-aversion parameter $\gamma$ and the proportional transaction costs $\lambda = \mu$.

As expected, the early exercise boundary moves up as both $\lambda = \mu$ and $\gamma$ increases, i.e., as both the proportional transaction costs and the risk-aversion increases.
CONCLUDING REMARKS AND POSSIBLE EXTENSIONS:

- WHAT ABOUT THE WRITER VIEWPOINT?
- DIFFERENT VOLATILITY DYNAMICS
- MORE EFFICIENT NUMERICAL SCHEMES
- ASYMPTOTIC EXPANSION FOR SMALL TIMES TO MATURETITY
THANKS FOR YOUR KIND ATTENTION!